Properties of a matrix norm: Let $A, B \in \mathbb{R}^{n\times n}$.

1. $\|A\| = 0$ *if, and only if,* $A = 0$ *in* $\mathcal{V}$;
2. $\|\lambda A\| = |\lambda|\,\|A\|$ *for all* $\lambda \in \mathbb{R}$ *and all* v *in* $\mathcal{V}$;
3. $\|A+B\| \le \|A\| + \|B\|$ *for all* u *and* v *in* $\mathcal{V}$ *(the triangle inequality).*
4. $\|Av\| \le \|A\|\,\|v\|, \qquad$ for all $v \in \mathbb{R}^n$.

Natural matrix norm induced by a vector norm.

Definition

$$\underbrace{\|A\|_p}_{\text{matrix p-norm}} = \max\left\{ \underbrace{\|Av\|_p}_{\text{vector p-norm}} : \underbrace{\|v\|_p}_{\text{vector p-norm}} = 1 \right\}.$$

Note $\|A\|_p$ satisfies ①, ② and ③ since it's defined in terms of vector norms which satisfy those properties...

Consider any vector $v \in \mathbb{R}^n$. Let $\hat{v} = \frac{v}{\|v\|_p}$ so $v = \hat{v}\,\|v\|_p$

scalar.. so by property 2

$$\|Av\|_p = \|A\hat{v}\,\|v\|_p\|_p = \|v\|_p \|A \underset{\text{unit vector}}{\hat{v}}\|_p \le \|v\|_p \underbrace{\max\{\|A\hat{v}\|_p : \|\hat{v}\|_p = 1\}}_{\|A\|_p}$$

Thus

$$\|Av\|_p \le \|A\|_p\,\|v\|_p$$

Note that any function $f: \mathbb{R}^{n\times n} \to \mathbb{R}$ that satisfies ①, ② ③ and ④ is a matrix norm. The induced or natural one (defined above), is the smallest M such that $\|Av\|_p \le M\,\|v\|_p$ holds for all $v \in \mathbb{R}^n$.

Let $A \in \mathbb{R}^{m \times n}$. $\|A\|_1 = \max\{\|Av\|_1 : \|v\|_1 = 1\}$.

Let $\|v\|_1 = 1$. Then

$$[Av]_i = \sum_{j=1}^{m} a_{ij} v_j$$

$$\|Av\|_1 = \sum_{i=1}^{m} \left|[Av]_i\right| = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} v_j \right|$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}||v_j| = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} |a_{ij}|\right)}_{\text{pull this out by taking its max}} |v_j|$$

Let $M = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}|\right)$

Then

$$\|Av\|_1 \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}|\right) |v_j| \leq \sum_{j=1}^{n} M |v_j| = M \sum_{j=1}^{n} |v_j| = M\|v\|_1 = M$$

So

$$\|A\|_1 = \max\{\|Av\|_1 : \|v\|_1 = 1\} \leq M$$

how sharp is this upper bound?

Need to estimate from below to find $\|A\|_1$ exactly.

$$M = \max_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}|\right) = \sum_{i=1}^{n} |a_{im}| \quad \text{for some } m.$$

note m depends on A but we're not changing A.

Note for any unit vector $v \in \mathbb{R}^n$ that $\|Av\|_1 \leq \|A\|_1$.

Let $e_m \in \mathbb{R}^n$ thus $e_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in the m^{th} coordinate.

$$\|e_m\|_1 = \sum_{j=1}^{n} [e_m]_j = 0+0+\cdots+1+0+\cdots+0 = 1$$

$$[Ae_m]_i = \sum_{j=1}^{n} a_{ij} \underbrace{[e_m]_j}_{m=j} =$$

Therefore

$$\|Ae_m\|_1 = \sum_{i=1}^{n} \left| [Ae_m]_i \right| = \sum_{i=1}^{n} |a_{im}| = M$$

Therefore

$$M = \|Ae_m\|_1 \leq \|A\|_1 \leq M$$

This means

$$\|A\|_1 = \max_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

Estimated below by choosing a particular unit vector.

Estimated above by using the triangle inequality.

The estimates were the same.

$$\|A\|_\infty = \max\{\|Av\|_\infty : \|v\|_\infty = 1\}.$$

$$[Av]_i = \sum_{j=1}^{n} a_{ij} v_j$$

$$\|Av\|_\infty = \max_{i=1}^{n} \left|[Av]_i\right| = \max_{i=1}^{n} \left|\sum_{j=1}^{n} a_{ij} v_j\right|$$

$$\leq \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \underbrace{|v_j|} \leq \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \left(\max_{l=1}^{n} |v_l|\right)$$

pull this term out by taking max

$$= \left(\max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\right) \|v\|_\infty = \underbrace{\max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|} = C$$

Let $C = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{mj}|$ for some m

Thus $\|A\|_\infty \leq C.$

$$|a_{mj}| = (\operatorname{sign} a_{mj})\, a_{mj} \qquad \text{where } \operatorname{sign} a_{mj} = \begin{cases} 1 & \text{if } a_{mj} > 0 \\ 0 & \text{if } a_{mj} = 0 \\ -1 & \text{if } a_{mj} < 0 \end{cases}$$

Let $v \in \mathbb{R}^n$ with $v_j = \operatorname{sign} a_{mj}$ Note $\|v\|_\infty = 1$

ignoring the special case when $A = 0$.

not triangle inequality.

$$\|Av\|_\infty = \max_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} v_j \right| \geq \left| \sum_{j=1}^{n} a_{mj} v_j \right| = \sum_{j=1}^{n} |a_{mj} v_j| = C$$

equality based on definition of V

Thus $\quad C = \|Av\|_\infty \leq \|A\|_\infty \leq C \quad$ so $\quad \|A\|_\infty = C.$

$$\|A\|_2 = \max \{ \|Av\|_2 : \|v\|_2 = 1 \}.$$

Recall, $\quad \|v\|_2^2 = v \cdot v \qquad \|Av\|_2^2 = Av \cdot Av$

$$\|A\|_2 = \max \{ \|Av\|_2^2 : \|v\|_2^2 = 1 \}^{1/2}.$$

Now

$$\|Av\|_2^2 = Av \cdot Av = A^T Av \cdot v = Bv \cdot v \quad \text{where } B = A^T A.$$

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$

Spectral Theorem

thus B is a symmetric matrix. Therefore there is an orthonormal basis of eigenvectors..

Thus

$$B\xi_i = \lambda_i \xi_i \quad \text{and} \quad \xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$Q = \begin{bmatrix} \xi_1 \,\Big|\, \xi_2 \,\Big|\, \cdots \,\Big|\, \xi_n \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} \xi_1^T \\ \hline \xi_2^T \\ \hline \vdots \\ \hline \xi_n^T \end{bmatrix} \begin{bmatrix} \xi_1 \,\Big|\, \xi_2 \,\Big|\, \cdots \,\Big|\, \xi_n \end{bmatrix} = \begin{bmatrix} \xi_1\cdot\xi_1 & \xi_1\cdot\xi_2 & \cdots & \xi_1\cdot\xi_n \\ \xi_2\cdot\xi_1 & & & \\ \vdots & & & \\ \xi_n\cdot\xi_1 & & & \xi_n\cdot\xi_n \end{bmatrix}$$

Therefore $Q^T Q = I$ since Q is square then $Q^{-1} = Q^T$.

$$BQ = B\begin{bmatrix} \xi_1 \,\Big|\, \xi_2 \,\Big|\, \cdots \,\Big|\, \xi_n \end{bmatrix} = \begin{bmatrix} B\xi_1 \,\Big|\, B\xi_2 \,\Big|\, \cdots \,\Big|\, B\xi_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\xi_1 \,\Big|\, \lambda_2\xi_2 \,\Big|\, \cdots \,\Big|\, \lambda_n\xi_n \end{bmatrix} = Q \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}}_{D} = QD$$

Thus $BQ = QD$ or $B = QDQ^{-1} = QDQ^T$

$$\|Av\|_2^2 = Av\cdot Av = A^TAv\cdot v = Bv\cdot v = QDQ^Tv\cdot v = D\underbrace{Q^Tv}\cdot Q^Tv$$

$c = Q^Tv$

$c = Q^{-1}v$

$v = Qc$

$$\|Av\|_2^2 = Dc\cdot c = \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\|v\|_2^2 = v\cdot v = Qc\cdot Qc = \underbrace{Q^TQ}_{I}c\cdot c = c\cdot c = \sum_{i=1}^{n} c_i^2$$

$$\|A\|_2 = \max\left\{ \|Av\|_2^2 : \|v\|_2^2 = 1 \right\}^{1/2}.$$

$$= \max\left\{ \underbrace{\sum_{i=1}^{n} \lambda_i c_i^2}_{} : \sum_{i=1}^{n} c_i^2 = 1 \right\}^{1/2}$$

weighted avg of the λ_i's
with weights c_i^2.