$$\|A\|_2 = \max\{\|Av\|_2^2 : \|v\|_2^2 = 1\}^{1/2}.$$

$$= \max\Big\{\underbrace{\sum_{i=1}^{n} \lambda_i c_i^2}_{\text{weighted avg of the } \lambda_i\text{'s with weights } c_i^2.} : \sum_{i=1}^{n} c_i^2 = 1\Big\}^{1/2}$$

Review

$\|Av\|^2 = Av\cdot Av = A^TAv\cdot v = Bv\cdot v$ where $B = A^TA$

spectral theorem gives eigenvalues and an orthonormal basis of eigenvectors

$B\xi_i = \lambda_i\xi_i$ and $\xi_i\cdot\xi_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}$

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ $Q = \begin{bmatrix} \xi_1 \,|\, \xi_2 \,|\, \cdots \,|\, \xi_n \end{bmatrix}$

$BQ = QD$ or $B = QDQ^T$

$\|Av\|^2 = QDQ^Tv\cdot v = D\underbrace{Q^Tv}_{c}\cdot Q^Tv = Dc\cdot c$ where $c = Q^Tv$

Since $c = Q^T v$ then $v = Qc$.

$$\|v\|^2 = v \cdot v = Qc \cdot Qc = \overbrace{Q^T Q}^{I} c \cdot c = c \cdot c = \sum_{i=1}^{n} c_i^2$$

$$\max\{\|Av\|_2^2 : \|v\|_2^2 = 1\}^{1/2} = \max\{Dc \cdot c : c \cdot c = 1\}^{1/2}$$

$$Dc = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}$$

$$Dc \cdot c = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\|Av\|_2 = \max\Big\{\sum_{i=1}^{n} \lambda_i c_i^2 : \sum_{i=1}^{n} c_i^2 = 1\Big\}^{1/2}$$

weighted average . . . of non-negative things.

Study the eigenvalues

$$\lambda_i \xi_i = B\xi_i = A^T A \xi_i$$

$$\lambda_i = \lambda_i \underbrace{\xi_i \cdot \xi_i}_{=1} = A^T A \xi_i \cdot \xi_i = A\xi_i \cdot A\xi_i = \|A\xi_i\|^2 \geq 0$$

Thus the eigenvalues are the norm squared of $A\xi_i$

↑ eigenvectors of B

Since the eigenvalues are real they are comparable and.

$$\min\{\lambda_i : i=1,\dots,n\} \leq \sum_{i=1}^{n} \lambda_i c_i^2 \leq \max\{\lambda_i : i=1,\dots,n\}$$

Therefore

$$\|Av\|_2 = \max\left\{\sum_{i=1}^{n} \lambda_i c_i^2 : \sum_{i=1}^{n} c_i^2 = 1\right\}^{1/2}$$

$$\leq \max\{\lambda_i : i=1,\dots,n\}^{1/2}$$

$$= \max\{\lambda_i^{1/2} : i=1,\dots,n\} = \max\{\sigma_i : i=1,\dots,n\}$$

where $\sigma_i = \lambda_i^{1/2}$ are called the singular values of A.

square root of eigenvalue of $B = A^T A$

Since $\lambda_i = \|A\xi_i\|^2$. So let m be the index so

$$\lambda_m^{1/2} = \max\{\lambda_i^{1/2} : i=1,\dots,n\}$$

Then $\lambda_m^{1/2} = \|A\xi_m\|$ and ξ_m is a unit vector..

$$\max\{\lambda_i^{1/2} : i=1,\dots,n\} = \|A\xi_m\| \leq \|A\|_2 \leq \max\{\lambda_i^{1/2} : i=1,\dots,n\}$$

Therefore $\|A\|_2 = \max\{\lambda_i^{1/2} : i=1,\dots,n\}$

Talk about estimating errors in the solution to $Ax=b$ by plugging in the approximation and seeing how well it satisfies the equation...

Setup let x be the exact solution

thus, $Ax=b$

let $\tilde{x} = x+\delta x$ be an approximate solution.

residual error $r = A\tilde{x} - b$

Let $\delta b = r$ then $A(x+\delta x) = b+\delta b$.

We have

$$Ax=b$$

$$A(x+\delta x) = b+\delta b$$

$$\cancel{Ax} + A\delta x = \cancel{b} + \delta b$$

Therefore $A\delta x = \delta b$ so $\delta x = A^{-1}\delta b$

And I can estimate the size of the error in x from the size of the residual error δb using matrix and vector norms.

$$\boxed{\|\delta x\|_p = \|A^{-1}\delta b\|_p \leq \|A^{-1}\|_p \|\delta b\|_p}$$

absolute error.

It is possible to approximate $\|A^{-1}\|_p$ without computing A^{-1}

What about Relative error $\frac{\|\delta x\|}{\|x\|}$. Can bound this?

$$Ax = b$$

$$\|b\| = \|Ax\| \le \|A\|\,\|x\|$$

$$\frac{1}{\|b\|} \ge \frac{1}{\|A\|\,\|x\|}$$

Multiply these inequalities together

Thus, $$\boxed{\frac{1}{\|x\|} \le \|A\| \frac{1}{\|b\|}}$$

$$\frac{\|\delta x\|}{\|x\|} \le \underbrace{\|A\|\,\|A^{-1}\|}_{\text{condition number of } A} \frac{\|\delta b\|}{\|b\|}$$

relative error in our approximation of x

relative residual error

Notation for condition number: $\kappa(A) = \|A\|\,\|A^{-1}\|$

Therefore:

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

Read the proof of

Theorem 2.10 *Given that* $\|\cdot\|$ *is a subordinate matrix norm on* $\mathbb{R}^{n\times n}$,

$$\|AB\| \leq \|A\|\,\|B\|$$

for any two matrices A *and* B *in* $\mathbb{R}^{n\times n}$.

over the weekend...

We just did this...

Definition 2.12 *The* **condition number** *of a nonsingular matrix* A *is defined by*

$$\kappa(A) = \|A\|\,\|A^{-1}\|\,.$$

Example...

2.8 Hilbert matrix $H = [h_{ij}]$ where $h_{ij} = \frac{1}{i+j-1}$

Try solving $Ha=b$

Interesting things... Condition number $\kappa(H)$ is large.

The solution to $Ha=b$ can be found exactly using algebra, etc.

```
help?> cond
search: cond condskeel Condition macroexpand @macroexpand

  cond(M, p::Real=2)

  Condition number of the matrix M, computed using the
  operator p-norm. Valid values for p are 1, 2 (default), or
  Inf.
```

by default $\|M\|_2 \|M^{-1}\|_2$

```
julia> cond(A)
14.933034373659268

julia> opnorm(A)*opnorm(inv(A))
14.93303437365925
```

almost the same...

Computing the same thing in different ways...

Can compute $\|A^{-1}\|$ without finding A^{-1}