Theorem 4.1 **(Contraction Mapping Theorem)** *Suppose that D is a closed subset of $\mathbb{R}^n$, $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined on D, and $\mathbf{g}(D) \subset D$. Suppose further that $\mathbf{g}$ is a contraction on D in the ∞-norm. Then, $\mathbf{g}$ has a unique fixed point $\boldsymbol{\xi}$ in D, and the sequence $(\boldsymbol{x}^{(k)})$ defined by (4.3) converges to $\boldsymbol{\xi}$ for any starting value $\boldsymbol{x}^{(0)} \in D$.*

means we can iterate

note contractions are continuous functions...

$$\|g(x)-g(y)\| \leq L\|x-y\| \quad \text{for some } L \in (0,1)$$

holds for all $x,y \in D$

Why do the iterates converge to a fixed point?

Let $x^{(0)} \in D$ then $x^{(1)} = g(x^{(0)}) \in D$ and $x^{(2)} = g(x^{(1)}) \in D$

$$\|x^{(2)} - x^{(1)}\| = \|g(x^{(1)}) - g(x^{(0)})\| \leq L\|x^{(1)} - x^{(0)}\|$$

Continue iterating $x^{(3)} = g(x^{(2)})$

$$\|x^{(3)} - x^{(2)}\| = \|g(x^{(2)}) - g(x^{(1)})\| \leq L\|x^{(2)} - x^{(1)}\| \leq L^2\|x^{(1)} - x^{(0)}\|$$

continue iterating

$\vdots$

$$\|x^{(k+2)} - x^{(k+1)}\| \leq L^{k+1}\|x^{(1)} - x^{(0)}\|$$

Simpler by shifting the index...

$$\|x^{(k+1)} - x^{(k)}\| \leq L^{k}\|x^{(1)} - x^{(0)}\|$$

$m=7$ $\quad$ $k=4$, $k+1=5$

$7 \quad 6 \quad 5$

3 pairs and $7-4=3$.

Suppose $m>k$. Then

$$\|x^{(m)} - x^{(k)}\| = \|\underbrace{x^{(m)} - x^{(m-1)}} + \underbrace{x^{(m-1)} - \cdots} + \cdots + \underbrace{x^{(k+1)} - x^{(k)}}\|$$

how many pairs of consecutive iterates are there?
$m-k$

by the triangle inequality

$$\| x^{(m)} - x^{(k)} \| \leq \| x^{(m)} - x^{(m-1)} \| + \| x^{(m-1)} - x^{(m-2)} \| + \cdots + \| x^{(k+1)} - x^{(k)} \|$$

$$\leq L^{(m-1)} \| x^{(1)} - x^{(0)} \| + L^{(m-2)} \| x^{(1)} - x^{(0)} \| + \cdots + L^{k} \| x^{(1)} - x^{(0)} \|$$

$$\leq \left(L^{k} + \cdots + L^{(m-1)} \right) \| x^{(1)} - x^{(0)} \|$$

$$\leq \left(\boxed{\sum_{j=0}^{\infty} L^{k+j}} \right) \| x^{(1)} - x^{(0)} \| = \frac{L^k}{1-L} \| x^{(1)} - x^{(0)} \|$$

simplify this ...

How to remember

$$\sum_{j=0}^{\infty} L^{k+j} = L^k + L^{k+1} + L^{k+2} + \cdots$$

$$L \sum_{j=0}^{\infty} L^{k+j} = L^{k+1} + L^{k+2} + L^{k+3} + \cdots$$

the cancellation here relies on $L \in (0,1)$ so the series converges

$$(1-L) \sum_{j=0}^{\infty} L^{k+j} = L^k$$

$$\sum_{j=1}^{\infty} L^{k+j} = \frac{L^k}{1-L}$$

In summary for $m > k$ ...

$$\| x^{(m)} - x^{(k)} \| \leq \frac{L^k}{1-L} \| x^{(1)} - x^{(0)} \|$$

since $L^k \to 0$ then

So given $\varepsilon > 0$ there is a k_0 such that $m, k \geq k_0$ implies

$$\| x^{(m)} - x^{(k)} \| < \varepsilon .$$

Such a sequence is called Cauchy.... the conclusion is that the iterates converge to some $\xi \in \mathbb{R}^n$.

Thus $\lim_{k\to\infty} x^{(k)} = \xi$.

Since $g(D) \subseteq D$ then all the iterates $x^{(k)} \in D$

since D is closed it contains all its limit points

Thus ξ is a limit point of $x^{(k)} \in D$ and D closed implies $\xi \in D$.

$$\xi = \lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty} g(x^{(k)}) = g\left(\lim_{k\to\infty} x^{(k)}\right) = g(\xi).$$

$k+1 \to \infty$ if and only if $k \to \infty$

interchange the limit with g because g is continuous at ξ

Therefore $g(\xi) = \xi$ and ξ is a fixed point.

Why are there no other fixed points in D ?

Suppose $\eta \neq \xi$ were such that $g(\eta) = \eta$.

$$\|\eta - \xi\| = \|g(\eta) - g(\xi)\| \leq L\|\eta - \xi\| \qquad \text{where } L \in (0,1)$$

$$\underbrace{(1-L)}_{\text{positive}}\ \underbrace{\|\eta - \xi\|}_{\text{positive if } \eta \neq \xi} \leq 0$$

Conclusion there is only one fixed point.

Do stuff with $\|x\|_\infty$ and $\|A\|_\infty$ next Monday...

matrix

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

$$\|A\|_\infty = C = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

Lab on Friday