The Computational Midterm is moved from Nov 27 to Dec 2.

Solutions of non-linear Systems

Theorem 4.1 (**Contraction Mapping Theorem**) *Suppose that D is a closed subset of $\mathbb{R}^n$, $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined on D, and $\mathbf{g}(D) \subset D$. Suppose further that $\mathbf{g}$ is a contraction on D in the ∞-norm. Then, $\mathbf{g}$ has a unique fixed point $\boldsymbol{\xi}$ in D, and the sequence $(\boldsymbol{x}^{(k)})$ defined by (4.3) converges to $\boldsymbol{\xi}$ for any starting value $\boldsymbol{x}^{(0)} \in D$.*

We just proved the above theorem. Let's use it!

1-step iteration schemes,

$x^{(k+1)} = g(x^{(k)})$ where $x^{(0)}$ is an initial approximation.

we know if g is a contraction and the other hypothesis of Theorem 4.1 hold then $x^{(k)}$ converges to a unique fixed point.

Theorem 4.2 *Suppose that $\mathbf{g} = (g_1, \ldots, g_n)^{\mathrm{T}}\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined and continuous on a closed set $D \subset \mathbb{R}^n$. Let $\boldsymbol{\xi} \in D$ be a fixed point of $\mathbf{g}$, and suppose that the first partial derivatives $\frac{\partial g_i}{\partial x_j}$, $j = 1, \ldots, n$, of g_i, $i = 1, \ldots, n$, are defined and continuous in some (open) neighbourhood $N(\boldsymbol{\xi}) \subset D$ of $\boldsymbol{\xi}$, with*

$$\boxed{\|Dg(\xi)\|_\infty = \|J_g(\boldsymbol{\xi})\|_\infty < 1.}$$

Then, there exists $\varepsilon > 0$ such that $\mathbf{g}(\bar{B}_\varepsilon(\boldsymbol{\xi})) \subset \bar{B}_\varepsilon(\boldsymbol{\xi})$, and the sequence defined by (4.3) converges to $\boldsymbol{\xi}$ for all $\boldsymbol{x}^{(0)} \in \bar{B}_\varepsilon(\boldsymbol{\xi})$.

Need to show that g satisfies the hypothesis of the perrous theorem "Contraction Mapping Theorem."

Need $D \subseteq \mathbb{R}^n$ closed such that $g(D) \subseteq D$ so I can iterate and stay in D.

also need g to be a contraction on D. There is $L \in (0,1)$ such that

$$\| g(x) - g(y) \|_\infty \leq L \, \| x - y \|_\infty \text{ for all } x, y \in D$$

Since $\| Dg(\xi) \|_\infty < 1$ and

$$Dg(x) = \begin{bmatrix} \partial g_1 / \partial x_1 & \partial g_1 / \partial x_2 & \cdots & \partial g_1 / \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_n / \partial x_1 & \partial g_n / \partial x_2 & \cdots & \partial g_n / \partial x_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is continuous. Choose $\varepsilon = \dfrac{1 - \| Dg(\xi) \|_\infty}{2}$

$\delta > 0$ such that

$$\| Dg(\xi) - Dg(x) \|_\infty < \varepsilon \quad \text{for all} \quad \| \xi - x \|_\infty < \delta.$$

Now

$$\| Dg(x) \|_\infty = \| Dg(x) - Dg(\xi) + Dg(\xi) \|_\infty$$

$$\leq \underbrace{\| Dg(x) - Dg(\xi) \|_\infty} + \underbrace{\| Dg(\xi) \|_\infty}_{\text{less than } 1.}$$

$$< \| Dg(\xi) \|_\infty + \varepsilon$$

$$\leq \| Dg(\xi) \|_\infty + \frac{1 - \| Dg(\xi) \|_\infty}{2}$$

$$= \underbrace{\frac{1 + \| Dg(\xi) \|_\infty}{2}}_{L} < 1$$

Therefore $\| x - \xi \|_\infty < \delta$ implies $\| Dg(x) \|_\infty \leq L$.

by continuity

$$\| x - \xi \|_\infty \leq \delta \text{ implies } \| Dg(x) \|_\infty \leq L.$$

Define $D = \{ x \in \mathbb{R}^n : \| x - \xi \|_\infty \leq \delta \}$.

Then D is closed and $x \in D$ implies $\| Dg(x) \|_\infty \leq L$

Also $\xi \in D$ and $g(\xi) = \xi$.

Need to show $g(D) \subseteq D$ and also that g is contraction on D.

Contraction means There is $L \in (0,1)$ such that

$$\|g(x)-g(y)\|_\infty \le L \boxed{\|x-y\|_\infty} \quad \text{for all } x,y \in D$$

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$\|g(x)-g(y)\|_\infty = \max \{ |g_i(x)-g_i(y)| : i=1,\dots,n \}.$$

Fix i and estimate $|g_i(x)-g_i(y)|$.

Let $x,y \in D$ then define $\varphi_i(t) = g_i(tx+(1-t)y)$.

Then $\varphi_i : [0,1] \to \mathbb{R}$ such that $\varphi_i(0) = g_i(y)$

$\varphi_i(1) = g_i(x)$

and

$$\varphi_i'(t) = \nabla g_i(tx+(1-t)y) \cdot \frac{d}{dt}(tx+(1-t)y)$$

$$= \nabla g_i(tx+(1-t)y) \cdot (x-y)$$

$$= \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\Big|_{x=(tx+(1-t)y)} (x_j - y_j)$$

By mean value theorem

$$g_i(x) - g_i(y) = \varphi_i(1) - \varphi_i(0) = \varphi_i'(\eta)\,(1-0)$$

for some $\eta \in (0,1)$

$$g_i(x) - g_i(y) = \sum_{j=1}^{n} \left.\frac{\partial g_i}{\partial x_j}\right|_{x=(\eta x + (1-\eta)y)} (x_j - y_j)$$

Since $\|x-y\|_\infty = \max\{|x_j - y_j| : j = 1, \dots, n\}$

$$|g_i(x) - g_i(y)| \le \sum_{j=1}^{n} \left|\frac{\partial g_i}{\partial x_j}(\eta x + (1-\eta)y)\right| \boxed{|x_j - y_j|}$$

$$\le \sum_{j=1}^{n} \left|\frac{\partial g_i}{\partial x_j}(\eta x + (1-\eta)y)\right| \|x-y\|_\infty$$

$$= \|x-y\|_\infty \sum_{j=1}^{n} \left|\frac{\partial g_i}{\partial x_j}(\eta x + (1-\eta)y)\right|$$

Therefore

$$|g_i(x) - g_i(y)| \le \|x-y\|_\infty \underbrace{\max\left\{ \sum_{j=1}^{n} \left|\frac{\partial g_i}{\partial x_j}(\eta x + (1-\eta)y)\right| : i = 1, \dots, n \right\}}_{\|Dg(\eta x + (1-\eta)y)\|_\infty}$$

Recall

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

Again for every i we have

$$\left|g_i(x) - g_i(y)\right| \leq \|x-y\|_\infty \, \|Dg(\eta x + (1-\eta)y)\|_\infty$$

$$\max\left\{\left|g_i(x) - g_i(y)\right| : i=1,\dots,n\right\} \leq \|x-y\|_\infty \, \|Dg(\eta x + (1-\eta)y)\|_\infty$$

Therefore

$$\|g(x)-g(y)\|_\infty \leq \|Dg(\eta x + (1-\eta)y)\|_\infty \, \|x-y\|_\infty$$

recall

$x \in D$ implies $\|Dg(x)\|_\infty \leq L$

Since $x, y \in D$ then $\eta \in (0,1)$ implies $\eta x + (1-\eta) y \in D$

Therefore $\|g(x)-g(y)\|_\infty \leq L \, \|x-y\|_\infty$ so g is a contraction.