$$A^{(k+1)} = R^{(pq)}(\varphi_k)^T A^{(k)} R^{(pq)}(\varphi_k)$$

where p and q are such that

$$|A^{(k)}_{pq}| = \max\{\,|A^{(k)}_{ij}| : i \neq j\,\}$$

and φ_k is chosen as in the $n=2$ case so $A^{(k+1)}_{pq} = 0$ with the additional requirement to choose t to be smallest in absolute value from among the two possible choices given by the quadratic equation.

did the same thing

```
function jacobi(A)
    p,q=getpq(A)
    t1,t2=quadsolv(-A[p,q],A[p,p]-A[q,q],A[p,q])
    t=t1
    if abs(t2)<abs(t1)
        t=t2
    end
    c=1/sqrt(1+t^2)
    s=c*t
    B=copy(A)
    B[p,p]=A[p,p]*c^2-2*A[p,q]*s*c+A[q,q]*s^2
    B[q,q]=A[p,p]*s^2+2*A[p,q]*s*c+A[q,q]*c^2
    B[p,q]=0
    B[q,p]=0
    N,_=size(A)
    for i=1:N
        if i==p||i==q
            continue
        end
        B[i,p]=A[i,p]*c-A[i,q]*s
        B[p,i]=B[i,p]
        B[i,q]=A[i,p]*s+A[i,q]*c
        B[q,i]=B[i,q]
    end
    return B
end
```

finds the largest in magnitude

```
julia> Ak=copy(A)
5×5 Matrix{Float64}:
  8.52  -1.8    1.08  -3.27  -3.25
 -1.8    1.76  -6.67  -2.67   4.05
  1.08  -6.67   1.16  -8.34   2.78
 -3.27  -2.67  -8.34   7.18   6.27
 -3.25   4.05   2.78   6.27  -3.28

julia> Ak=jacobi(Ak)
5×5 Matrix{Float64}:
  8.52       -1.8       -0.995368  -3.29675  -3.25
 -1.8         1.76      -6.99297    1.64807   4.05
 -0.995368   -6.99297   -4.69655    0.0       5.87835
 -3.29675     1.64807    0.0       13.0365    3.5336
 -3.25        4.05       5.87835    3.5336   -3.28
```

Perform an orgothonal transformation using a plane rotation to put zeros where the largest was.

$|A_{34}|$ was the largest

p q

p q

q

the rest dont change

these rows and columns change

Today we show that $A^{(k)} \to D$ where D is a diagonal matrix with the eigenvalues of A.

The Trace of a matrix (The sum of the diagonal terms)

Let $A \in \mathbb{R}^{n \times n}$ then $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$

The spectral Norm of a Matrix

$$\|A\|_2 = \max\{\lambda_i^{1/2} \text{ where } \lambda_i \text{ are the eigenvalues of } B = A^TA\}$$

Frobenius Norm

$$\|A\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2}$$

idea is treat the elements of A as if they were a vector and take the 2-vector Norm

$$\text{Trace } B = \sum_{i=1}^{n} [A^T A]_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ki} A_{ki}$$

$$[A^T A]_{ij} = \sum_{k=1}^{n} [A^T]_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj}$$

$$\text{Trace } B = \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2 \quad \text{since } A \in \mathbb{R}^{n \times n}$$

Therefore

$$\boxed{\|A\|_F^2 = \text{trace}(A^T A) \quad \text{for } A \in \mathbb{R}^{n \times n}.}$$

<u>Properties of trace</u> $A, B \in \mathbb{R}^{n \times n}$

$$\text{trace } AB = \sum_{i=1}^{n} [AB]_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$[AB]_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

the same

$$\text{trace } BA = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ik} A_{ki}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ki} A_{ik} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

Therefore trace AB = trace BA.

$$A^{(k+1)} = R^{(pq)}(\varphi_k)^T A^{(k)} R^{(pq)}(\varphi_k)$$

$$A^{(k+1)} = R^{(pq)}(\varphi_k)^{-1} A^{(k)} R^{(pq)}(\varphi_k)$$

Consider $B = R^{-1} A R$ where R is an orthogonal matrix.

$$\text{trace } B = \text{trace } \underbrace{R^{-1}}_{A} \underbrace{A R}_{B} = \text{trace } A R R^{-1} = \text{trace } A$$

Therefore trace A = trace $A^{(1)}$ = trace $A^{(2)}$ = ... = trace $A^{(k)}$

assume A is symmetric

$A^T = A$

$$\|A\|_F^2 = \text{trace}(A^T A) = \text{trace}(A^2)$$

$$\|B\|_F^2 = \text{trace}(B^T B) = \text{trace}(R^T A R)^T R^T A R$$

$$= \text{trace } R^T A^T R^{TT} R^T A R = \text{trace } R^T A^2 R$$

symmetric

cancel

$$= \text{trace } R^{-1} A^2 R = \text{trace } A^2 R R^{-1} = \text{trace } A^2 = \|A\|_F^2$$

Lemma 5.1 *The sum of squares of the elements of a symmetric matrix is invariant under an orthogonal transformation: that is, if* $A \in \mathbb{R}^{n\times n}_{\text{sym}}$

Theorem 5.3 *Suppose that $A \in \mathbb{R}^{n\times n}_{\text{sym}}$, $n \geq 2$. In the classical Jacobi method the off-diagonal entries in the sequence of matrices $(A^{(k)})$, generated from $A^{(0)} = A$ according to Definition 5.3, converge to 0 in the sense that*

$$\lim_{k\to\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} [(A^{(k)})_{ij}]^2 = 0\,. \tag{5.12}$$

Furthermore,

$$\lim_{k\to\infty} \sum_{i=1}^{n} [(A^{(k)})_{ii}]^2 = \text{Trace}(A^2)\,. \tag{5.13}$$

Note once we know $A^{(k)} \to D$ and D is diagonal its automatic that the diagonal entrie of are the eigenvalues of A.

$$A^{(k+1)} = R^T A^{(k)} R = \underbrace{R^{-1} A^{(k)} R}$$

similarity doesn't change the eigenvalues.

The eigen values are the roots of $\chi_A(\lambda) = \det(A - \lambda I)$

$$B = R^{-1} A R$$

$$\chi_B(\lambda) = \det(B - \lambda I) = \det(R^{-1} A R - \lambda I)$$

$$= \det(R^{-1} A R - \lambda R^{-1} R)$$ now factor inside

$$= \det\left((R^{-1}A - \lambda R^{-1}) R\right) = \det(\underbrace{R^{-1}}_{A} \underbrace{(A - \lambda I) R}_{B})$$

$$= \det\left((A - \lambda I) \cancel{R} \cancel{R^{-1}}\right) = \det(A - \lambda I) = \chi_A(\lambda)$$

It follows that the eigenvalues of $A^{(k+1)}$ are the same as $A^{(k)}$ and so forth.. so in the limit the eigenvalues of D are the same as A.

We know $\| A^{(k+1)} \|_F^2 = \| A^{(k)} \|_F^2$

Let $B = R^T A R$ where $A^T = A$ and $R^T = R^{-1}$

and $R = R^{(pq)}(\varphi_k)$ (plane rotation)

$$\|B\|_F^2 = \sum_{i,j=1}^{n} |B_{ij}|^2 = \sum_{i=1}^{n} |B_{ii}|^2 + \sum_{i \neq j} |B_{ij}|^2$$

$$\|$$

$$\|A\|_F^2 = \sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i=1}^{n} |A_{ii}|^2 + \sum_{i \neq j} |A_{ij}|^2$$

Thus

$$\underbrace{\sum_{i=1}^{n} |B_{ii}|^2}_{D(B)} + \underbrace{\sum_{i \neq j} |B_{ij}|^2}_{L(B)} = \underbrace{\sum_{i=1}^{n} |A_{ii}|^2}_{D(A)} + \underbrace{\sum_{i \neq j} |A_{ij}|^2}_{L(A)}$$

```
B[p,p]=A[p,p]*c^2-2*A[p,q]*s*c+A[q,q]*s^2
B[q,q]=A[p,p]*s^2+2*A[p,q]*s*c+A[q,q]*c^2
```

Thinking about the sub matrices

$$\begin{bmatrix} b_{pp} & b_{pq} \\ b_{qp} & b_{qq} \end{bmatrix} = R^T \begin{bmatrix} a_{pp} & a_{pq} \\ a_{qp} & a_{qq} \end{bmatrix} R$$

where

$$R = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.$$

again

$$\left\| \begin{bmatrix} b_{pp} & b_{pq} \\ b_{qp} & b_{qq} \end{bmatrix} \right\|_F = \left\| \begin{bmatrix} a_{pp} & a_{pq} \\ a_{qp} & a_{qq} \end{bmatrix} \right\|_F$$

these are zero

Therefore

$$a_{pp}^2 + a_{qq}^2 + 2a_{qp} = b_{pp}^2 + b_{qq}^2 + \cancelto{0}{2b_{qp}}$$

finish next time ...