Let $A \in \mathbb{R}^{n \times n}$ with $A^T = A$ and $B = R^T A R$.

where $R = R^{(pq)}(\phi)$ so that p and q are given by

$$|a_{pq}| = \max\{|a_{ij}| : i \neq j\}$$

and ϕ is chosen so $b_{pq} = 0$.

From last time we have

recall $\|A\|_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}$

$$\|A\|_F^2 = \|B\|_F^2$$

also

$$\left\| \begin{bmatrix} a_{pp} & a_{pq} \\ a_{pq} & a_{qq} \end{bmatrix} \right\|_F^2 = \left\| \begin{bmatrix} b_{pp} & 0 \\ 0 & b_{qq} \end{bmatrix} \right\|_F^2$$

Substitute the second into the first...

$$a_{pp}^2 + 2a_{pq}^2 + a_{qq}^2 = b_{pp}^2 + b_{qq}^2$$

$$\sum_{i,j} a_{ij}^2 = \sum_{i,j} b_{ij}^2$$

substitute

$$\sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j} a_{ij}^2 = \sum_{i=1}^{n} b_{ii}^2 + \sum_{i \neq j} b_{ij}^2$$

Since only the pp and qq entries of the diagonal are affected by this rotation

$$\sum_{i=1}^{n} a_{ii}^2 - a_{pp}^2 - a_{qq}^2 = \sum_{i=1}^{n} b_{ii}^2 - b_{pp}^2 - b_{qq}^2$$

equivalently

$$\sum_{i=1}^{n} a_{ii}^2 + b_{pp}^2 + b_{qq}^2 = \sum_{i=1}^{n} b_{ii}^2 + a_{pp}^2 + a_{qq}^2$$

After substituting

$$\sum_{i=1}^{n} a_{ii}^2 + \cancel{a_{pp}^2} + 2a_{pq}^2 + \cancel{a_{qq}^2} = \sum_{i=1}^{n} b_{ii}^2 + \cancel{a_{pp}^2} + \cancel{a_{qq}^2}$$

$$\sum_{i=1}^{n} b_{ii}^2 = \sum_{i=1}^{n} a_{ii}^2 + 2a_{pq}^2$$

So the diagonal increases by $2a_{pq}^2$ and the sum of all the squares is unchanged..

Therefore the off diagonal terms decrease by the same amount...

$$\underbrace{\sum_{i\neq j} b_{ij}^2} = \sum_{i\neq j} a_{ij}^2 - 2a_{pq}^2$$

recall

$$|a_{pq}| = \max\{|a_{ij}| : i \neq j\}$$

$$L(B) = \sum_{i\neq j} b_{ij}^2$$

Thus

$$L(B) = L(A) - 2a_{pq}^2$$

$$L(A) = \sum_{i\neq j} a_{ij}^2 \leq \sum_{i\neq j} a_{pq}^2 = n(n-1)\, a_{pq}^2$$

constant

$$-a_{pq}^2 \leq \frac{-1}{n(n-1)} L(A)$$

It follows

$$L(B) \leq L(A) - \frac{2}{n(n-1)} L(A) = \underbrace{\left(1 - \frac{2}{n(n-1)}\right)} L(A)$$

note since $n \geq 2$

then $1 - \frac{2}{n(n-1)} < 1$

In general

$$L(A^{(k+1)}) \leq \left(1 - \frac{2}{n(n-1)}\right) L(A^{(k)})$$

Therefore

$$h(A^{(k)}) \le \left(1 - \frac{2}{n(n-1)}\right)^k h(A^{(0)}) = \left(1 - \frac{2}{n(n-1)}\right)^k h(A)$$

Taking limits implies $h(A^{(k)}) \to 0$ and so $A^{(k)}$ converges to a diagonal matrix as $k \to \infty$.

<u>Remark</u>: Since $h(A^{(k)})$ represents how far away $A^{(k)}$ is from being a diagonal matrix then we can view $h(A^{(k)})$ as error in our approximation of the eigenvalues... Then

$$h(A^{(k+1)}) \le \left(1 - \frac{2}{n(n-1)}\right) h(A^{(k)})$$

implies the Jacobi's method for finding eigenvalues converges at least linearly. In practice it's faster...

```
julia> include("jeigen.jl")
k=1 LAk=281.44680000000005 logLAk/logLAkm1=0.9335209002990803
k=2 LAk=183.64349659829549 logLAk/logLAkm1=0.9242994035618817
k=3 LAk=81.84721263489874 logLAk/logLAkm1=0.8449755082765302
k=4 LAk=54.36574538954163 logLAk/logLAkm1=0.907120655895368
k=5 LAk=15.929254090080738 logLAk/logLAkm1=0.6927781248585756
k=6 LAk=10.15393223646082 logLAk/logLAkm1=0.8373299610158177
k=7 LAk=5.964736853566595 logLAk/logLAkm1=0.7704797252420562
k=8 LAk=3.318218794089802 logLAk/logLAkm1=0.6716230912110123
k=9 LAk=0.9977319229716877 logLAk/logLAkm1=-0.0018931130192760188
k=10 LAk=0.59351346849735 logLAk/logLAkm1=229.75565591715488
k=11 LAk=0.20650037503999064 logLAk/logLAkm1=3.023705279989643
k=12 LAk=0.03672604066633336 logLAk/logLAkm1=2.0946862537304987
k=13 LAk=0.011645890117844643 logLAk/logLAkm1=1.3475905392458067
k=14 LAk=0.007008076978792447 logLAk/logLAkm1=1.140607659960113
k=15 LAk=0.003947994623412041 logLAk/logLAkm1=1.1156805512411085
k=16 LAk=0.0017163072686254594 logLAk/logLAkm1=1.150515053093877
k=17 LAk=0.00019462109920937416 logLAk/logLAkm1=1.3418685996856967
k=18 LAk=2.5035339939605697e-5 logLAk/logLAkm1=1.2400113208079726
k=19 LAk=8.74127069130712e-6 logLAk/logLAkm1=1.099312014780328
k=20 LAk=3.1850843467390176e-6 logLAk/logLAkm1=1.0866778877229943
k=21 LAk=1.0316051320207014e-7 logLAk/logLAkm1=1.2709914893244203
k=22 LAk=8.943357088687745e-11 logLAk/logLAkm1=1.4382765099219976
```

rate slow.

here the rate of convergence looks bigger than 1 pg 145

Definition 5.5 *Suppose that* $n \geq 2$ *and* $A \in \mathbb{C}^{n\times n}$. *The* **Gerschgorin discs** D_i, $i = 1, 2, \ldots, n$, *of the matrix* A *are defined as the closed circular regions*

$$D_i = \{z \in \mathbb{C}\colon |z - a_{ii}| \leq R_i\} \tag{5.17}$$

in the complex plane, where

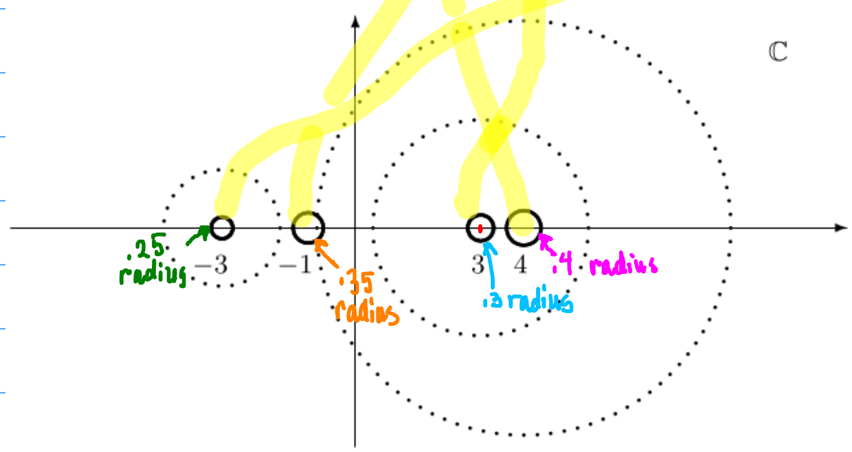
$$R_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \tag{5.18}$$

sum over the rest of the row

is the radius of D_i.

Example 5.4 *Consider the matrix*

$$A = \begin{array}{cc} \begin{pmatrix} 4.00 & 0.20 & -0.10 & 0.10 \\ 0.20 & -1.00 & -0.10 & 0.05 \\ -0.10 & -0.10 & 3.00 & 0.10 \\ 0.10 & 0.05 & 0.10 & -3.00 \end{pmatrix} & \begin{matrix} .4 \\ .35 \\ .3 \\ .25 \end{matrix} \end{array}$$



Let x be an eigenvector with eigenvalue λ.

$A x = \lambda x$

Need to show λ is in one of the discs.

Let k be such that $|x_k| = \max\{|x_i| : i = 1, .., n\}$.