Definition 5.5 *Suppose that $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$. The* **Gerschgorin discs** *D_i, $i = 1, 2, \ldots, n$, of the matrix A are defined as the closed circular regions*

$$D_i = \{z \in \mathbb{C}\colon |z - a_{ii}| \leq R_i\} \tag{5.17}$$

discs in complex plane

in the complex plane, where

$$R_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \tag{5.18}$$

of diagonal terms

is the radius of D_i.

Theorem 5.4 (Gerschgorin's Theorem) *Let $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$. All eigenvalues of the matrix A lie in the region $D = \bigcup_{i=1}^{n} D_i$, where D_i, $i = 1, 2, \ldots, n$, are the Gerschgorin discs of A defined by (5.17), (5.18).*

Theorem 5.5 (Gerschgorin's Second Theorem) *Let $n \geq 2$. Suppose that $1 \leq p \leq n-1$ and that the Gerschgorin discs of the matrix $A \in \mathbb{C}^{n\times n}$ can be divided into two disjoint subsets $D^{(p)}$ and $D^{(q)}$, containing p and $q = n - p$ discs respectively. Then, the union of the discs in $D^{(p)}$ contains p of the eigenvalues, and the union of the discs in $D^{(q)}$ contains $n - p$ eigenvalues. In particular, if one disc is disjoint from all the others, it contains exactly one eigenvalue, and if all the discs are disjoint then each disc contains exactly one eigenvalue.*

Idea is that an iterative scheme has been used to create a matrix that's approximately diagonal. What the error in the approximation of the eigenvalues

Proof: Let x be an eigenvector and λ the corresponding eigenvalue. Then $Ax = \lambda x$. Need to show $\lambda \in \bigcup_{i=1}^{n} D_i$

Since $x \neq 0$ let k to be so $|x_k| = \max\{|x_i| : i = 1, \ldots, n\}$.

$$|\lambda - a_{kk}||x_k| = |\lambda x_k - a_{kk} x_k|$$

Since $Ax = \lambda x$ then $[Ax]_k = \underbrace{[\lambda x]_k}_{\lambda x_k}$

The k^{th} components are equal

$$[Ax]_k = \sum_{j=1}^{n} a_{kj} x_j$$

Therefore

cancel that term from the sum

$$|\lambda - a_{kk}||x_k| = |\lambda x_k - a_{kk} x_k| = \left| \sum_{j=1}^{n} a_{kj} x_j - a_{kk} x_k \right|$$

$$= \left| \sum_{j \neq k} a_{kj} x_j \right| \leq \sum_{j \neq k} |a_{kj}| \underbrace{|x_j|}_{\text{max}} \leq \left(\sum_{j \neq k} |a_{kj}| \right) |x_k|$$

recall.

$$\leq R_k |x_k|$$

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

$$D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq R_i \}$$

Therefore

$$|\lambda - a_{kk}| \cancel{|x_k|} \leq R_k \cancel{|x_k|}$$

So $|\lambda - a_{kk}| \leq R_k$ thus $\lambda \in D_k$

Therefore all λ eigenvalues are in $\bigcup_{i=1}^{n} D_i$.