2.1 Let $n \geq 2$. Given the matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, the permutation matrix $Q \in \mathbb{R}^{n \times n}$ reverses the order of the rows of A, so that $(QA)_{i,j} = a_{n+1-i,j}$. If $L \in \mathbb{R}^{n \times n}$ is a lower triangular matrix, what is the structure of the matrix QLQ?

Show how to factorise $A \in \mathbb{R}^{n \times n}$ in the form $A = UL$, where $U \in \mathbb{R}^{n \times n}$ is unit upper triangular and $L \in \mathbb{R}^{n \times n}$ is lower triangular. What conditions on A will ensure that the factorisation exists? Give an example of a square matrix A which cannot be factorised in this way.

Since Q is a permutation matrix, then it is orthogonal. Thus Q^{-1} exists and $Q^{-1} = Q^T$. Since reversing the rows twice is the identity operation, then $Q^{-1} = Q$. It follows that $Q^T = Q$ so the matrix Q is symmetric.

First not that L^T is upper triangur. Thus

$$(LQ)^T = Q^T L^T = Q L^T$$

is an upper triangular matrix with the order of the rows reversed. Diagramatically it look like

$$L^T = \begin{bmatrix} x & x & \cdots & x \\ & x & & \vdots \\ & 0 & \ddots & \vdots \\ & & & x \end{bmatrix} \qquad QL^T = \begin{bmatrix} 0 & & & x \\ & & \cdot^{\cdot^{\cdot}} & \vdots \\ & x & & \vdots \\ x & x & \cdots & x \end{bmatrix}$$

where the x's represent possibly non-zero entries. Transposing this matrix does not change the structure, thus

$$LQ = (LQ)^{TT}$$

Also looks like

$$LQ = \begin{bmatrix} 0 & & & x \\ & & \iddots & \vdots \\ & x & & \vdots \\ x & x & \cdots & x \end{bmatrix}$$

Finally QLQ is obtained by reversing the rows of LQ to obtain a matrix that looks like

$$QLQ = \begin{bmatrix} x & x & \cdots & x \\ & x & & \vdots \\ & & \ddots & \vdots \\ 0 & & & x \end{bmatrix}$$

which is upper triangular.

To see how to factorize $A = UL$ consider A^T and factor this matrix so $QAQ = LU$ using the usual method by Gaussian elimination. This method succeeds if no row swaps need to be made while performing the elimination steps.

Since $QQ = I$ it follows that

$$A = QLUQ = QL\,I\,UQ = QL(QQ)UQ = (QLQ)(QUQ)$$

By what we've just shown QLQ is upper triangular. Similar reasoning implies QUQ is lower triangular. Setting

$$\tilde{U} = QLQ \quad \text{and} \quad \tilde{L} = QUQ$$

now yields the factorization $A = \tilde{U}\tilde{L}$ where $\tilde{U}$ is upper triangular and $\tilde{L}$ is lower triangular.

Suppose $QAQ = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Then there is a 0 in the pivot position so QAQ can't be factored as LU.

Noting that $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and solving back for A yields

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

It follows that $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ can't be factored as $\tilde{U}\tilde{L}$.

2.8 (i) Show that, for any vector $\boldsymbol{v} = (v_1, \ldots, v_n)^{\mathrm{T}} \in \mathbb{R}^n$,

$$\|\boldsymbol{v}\|_\infty \le \|\boldsymbol{v}\|_2 \text{ and } \|\boldsymbol{v}\|_2^2 \le \|\boldsymbol{v}\|_1 \|\boldsymbol{v}\|_\infty .$$

In each case give an example of a nonzero vector $\boldsymbol{v}$ for which equality is attained. Deduce that $\|\boldsymbol{v}\|_\infty \le \|\boldsymbol{v}\|_2 \le \|\boldsymbol{v}\|_1$. Show also that $\|\boldsymbol{v}\|_2 \le \sqrt{n}\, \|\boldsymbol{v}\|_\infty$.

(ii) Show that, for any matrix $A \in \mathbb{R}^{m \times n}$,

$$\|A\|_\infty \le \sqrt{n}\, \|A\|_2 \text{ and } \|A\|_2 \le \sqrt{m}\, \|A\|_\infty .$$

In each case give an example of a matrix A for which equality is attained. (See the footnote following Definition 2.12 for the meaning of $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$ when $A \in \mathbb{R}^{m \times n}$.)

(i) Let $v \in \mathbb{R}^n$. Now for each i we have

$$|v_i|^2 \le |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2 .$$

Thus

$$|v_i| \le \sqrt{|v_1|^2 + \cdots + |v_n|^2} = \|v\|_2 .$$

Then

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$$
$$\le \max\{\|v\|_2, \|v\|_2, \ldots, \|v\|_2\} = \|v\|_2$$

Therefore $\|v\|_\infty \le \|v\|_2$.

To see the other inequality estimate

$$\|v\|_2^2 = |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2 = \sum_{i=1}^{n} |v_i|\,|v_i|$$

$$\leq \sum_{i=1}^{n} \max\{|v_1|, |v_2|, \ldots, |v_n|\}\ |v_i| = \sum_{i=1}^{n} \|v\|_\infty |v_i|$$

$$= \|v\|_\infty \sum_{i=1}^{n} |v_i| = \|v\|_\infty \|v\|_1.$$

Therefore

$$\|v\|_2^2 \leq \|v\|_\infty \|v\|_1.$$

Since $\|v\|_\infty \leq \|v\|_2$ from before, substituting yields

$$\|v\|_2^2 \leq \|v\|_\infty \|v\|_1 \leq \|v\|_2 \|v\|_1.$$

Now if $\|v\|_2 = 0$ then $v = 0$ and clearly

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1$$

If $\|v\|_2 \neq 0$ we can cancel and obtain $\|v\|_2 \leq \|v\|_1$. In this case it again follows that

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1.$$

Finally we show that $\|v\|_2 \leq \sqrt{n}\,\|v\|_\infty$. To see this note that

$$\|v\|_2^2 = \sum_{i=1}^{n} |v_i|^2 \leq \sum_{i=1}^{n} \max\{|v_1|^2, |v_2|^2, \dots, |v_n|^2\}$$

$$= \sum_{i=1}^{n} \|v\|_\infty^2 = n\|v\|_\infty^2.$$

Taking square roots now yields that

$$\|v\|_2 \leq \sqrt{n}\|v\|_\infty.$$

(ii) Let $A \in \mathbb{R}^{m \times n}$. By definition

$$\|A\|_\infty = \max\{\|Ax\|_\infty : \|x\|_\infty = 1\}$$

$$\leq \max\{\|Ax\|_2 : \|x\|_\infty = 1\}$$

since the vector norms $\|Ax\|_\infty \leq \|Ax\|_2$. Now

$$\|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

implies

$$\{x : \|x\|_\infty = 1\} \subseteq \{x : \|x\|_2 \leq \sqrt{n}\}$$

Therefore taking the maximum over a larger set yields

$$\|A\|_\infty \leq \max\{\|Ax\|_2 : \|x\|_2 \leq \sqrt{n}\}$$

On the other hand

$$\|A\|_2 = \max\{\|Ax\|_2 : \|x\|_2 = 1\}$$

Now if $0 < \|x\|_2 \leq 1$ then $\frac{x}{\|x\|_2} = 1$ and

$$\|Ax\|_2 = \|Ax\|_2 \frac{\|x\|_2}{\|x\|_2} = \left\| A \frac{x}{\|x\|_2} \right\|_2 \|x\|_2$$

$$\leq \|A\|_2 \|x\|_2 \leq \|A\|_2$$

Consequently

$$\max\{\|Ax\|_2 : \|x\|_2 \leq 1\} \leq \|A\|_2$$

and so

$$\|A\|_2 = \max\{\|Ax\|_2 : \|x\|_2 \leq 1\}.$$

To finish note that $\|Ax\|_2 \frac{\sqrt{n}}{\sqrt{n}} = \sqrt{n} \|A \frac{x}{\sqrt{n}}\|_2$. Thus

$$\max\{\|Ax\|_2 : \|x\|_2 \leq 1\} = \max\{\tfrac{1}{\sqrt{n}} \|A\sqrt{n}x\|_2 : \|x\|_2 \leq 1\}$$

and taking $y = \sqrt{n}x$ yields $x = \frac{y}{\sqrt{n}}$ so that

$$\|A\|_2 = \max\left\{\tfrac{1}{\sqrt{n}} \|Ay\|_2 : \left\|\tfrac{y}{\sqrt{n}}\right\|_2 \leq 1\right\}$$

$$= \frac{1}{\sqrt{n}} \max\{\|Ay\|_2 : \|y\|_2 \leq \sqrt{n}\}$$

Combining this with the bound for $\|A\|_\infty$ yield

$$\|A\|_\infty \leq \max\{\|Ax\|_2 : \|x\|_2 \leq \sqrt{n}\} = \sqrt{n}\|A\|_2$$

as desired.

Now bound $\|A\|_2$ in terms of $\|A\|_\infty$. Since $Ax \in \mathbb{R}^m$ then part (i) shows $\|Ax\|_2 \le \sqrt{m}\,\|Ax\|_\infty$. It follows that

$$\|A\|_2 = \max\{\|Ax\|_2 : \|x\|_2 = 1\} \le \max\{\sqrt{m}\,\|Ax\|_\infty : \|x\|_2 = 1\}$$

Now $\|x\|_\infty \le \|x\|_2$ implies

$$\{x : \|x\|_2 = 1\} \subseteq \{x : \|x\|_\infty \le 1\}$$

therefore, taking maximum over the larger set yields

$$\max\{\sqrt{m}\,\|Ax\|_\infty : \|x\|_2 = 1\} \le \max\{\sqrt{m}\,\|Ax\|_\infty : \|x\|_\infty \le 1\}.$$

As before $\|x\|_\infty \le 1$ implies $\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty \le \|A\|_\infty$ so

$$\|A\|_\infty = \max\{\|Ax\|_\infty : \|x\|_\infty = 1\} = \max\{\|Ax\|_\infty : \|x\|_\infty \le 1\}$$

Consequently

$$\|A\|_2 \le \max\{\sqrt{m}\,\|Ax\|_\infty : \|x\|_\infty \le 1\}$$

$$= \sqrt{m}\max\{\|Ax\|_\infty : \|x\|_\infty \le 1\} = \sqrt{m}\,\|A\|_\infty.$$

which was to be shown.