$$\boxed{f(x) = x+1}$$

cancellation for x close to -1

$$\tilde{f}(x) = fl(fl(x)+1)$$

$fl(x) = x + \varepsilon x$ where $|\varepsilon| \le \frac{1}{2}\varepsilon_{mach}$

Simplifying assumption that adding 1 can be done exactly...

$$fl(x)+1 \in \mathbb{F}$$

Thus no generated error from the $\oplus$.

$$\text{relative error} = \frac{|\tilde{f}(x) - f(x)|}{|f(x)|} = \frac{|fl(x)+1-(x+1)|}{|x+1|} =$$

$$= \frac{|x+\varepsilon x + 1 - x - 1|}{|x+1|} = \frac{\varepsilon|x|}{|x+1|}$$

Note: if x is close to -1 then $x+1$ is close to 0 and the relative error is really big!

Loss of precision due to subtracting two nearly equal numbers..

Condition number: $f: \mathbb{R} \to \mathbb{R}$ $\quad \tilde{x} = fl(x)$

$$\left|\frac{f(\tilde{x}) - f(x)}{f(x)}\right| \Big/ \left|\frac{\tilde{x} - x}{x}\right|$$

relative error in the output of f with respect to relative error in the input

What happens in the limit of better and better approximations of x by $\tilde{x}$.

Suppose $\tilde{x} = x + \varepsilon x$ where $0 < \varepsilon < \frac{1}{2}\varepsilon_{mach}$.

So denominator is not zero

$$\frac{\left|\frac{f(\tilde{x})-f(x)}{f(x)}\right|}{\left|\frac{\tilde{x}-x}{x}\right|} = \frac{\left|\frac{f(x+\varepsilon x)-f(x)}{f(x)}\right|}{\left|\frac{\cancel{x}+\varepsilon x-\cancel{x}}{x}\right|}$$

small change

$$= \left|\frac{f(x+\varepsilon x)-f(x)}{\varepsilon f(x)}\right| = \left|\frac{f(x+\varepsilon x)-f(x)}{\varepsilon x}\right|\left|\frac{x}{f(x)}\right|$$

$$\longrightarrow \underbrace{|f'(x)|\left|\frac{x}{f(x)}\right|}_{K_f(x)} \quad \text{as } \varepsilon \to 0$$

$K_f(x)$ relative condition number of f at x

Since

$$\frac{\left|\frac{f(\tilde{x})-f(x)}{f(x)}\right|}{\left|\frac{\tilde{x}-x}{x}\right|} \longrightarrow K_f(x) \quad \text{as } \varepsilon \to 0$$

then if $\varepsilon \approx 0$ then

$$\frac{\left|\frac{f(\tilde{x})-f(x)}{f(x)}\right|}{\left|\frac{\tilde{x}-x}{x}\right|} \approx K_f(x)$$

biggest is $\frac{1}{2}\varepsilon_{mach}$

Therefore....

$$\left|\frac{f(\tilde{x}) - f(x)}{f(x)}\right| \lesssim \frac{1}{2}\varepsilon_{mach} K_f(x)$$

fixed

Suppose $f(a)$ means solve $ax^2 + bx + c = 0$ for x

parameter

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2}$$

To make specific. $f(a) = r_1$. Condition of the root r_1 on the parameter a.

Implicit differentiation...

$$\frac{d}{da}\left(a\underbrace{r_1^2} + br_1 + c\right) = \frac{d}{da}0$$

$$\left(\frac{da}{da}\right)r_1^2 + a\frac{dr_1^2}{da} + b\frac{dr_1}{da} = 0$$

$$r_1^2 + 2ar_1\frac{dr_1}{da} + b\frac{dr_1}{da} = 0$$

$$\frac{dr_1}{da} = \frac{-r_1^2}{2ar_1 + b} = \frac{-r_1^2}{a(r_1 - r_2)} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$2ar_1 + b = \sqrt{b^2 - 4ac} = a(r_1 - r_2)$$

$$2ar_1 + b = \sqrt{b^2 - 4ac}$$

$$r_1 - r_2 = \frac{\cancel{-b} + \sqrt{b^2 - 4ac}}{2a} - \frac{\cancel{-b} - \sqrt{b^2 - 4ac}}{2a} = \frac{\cancel{2}\sqrt{b^2 - 4ac}}{\cancel{2}a}$$

$$a(r_1 - r_2) = \sqrt{b^2 - 4ac}$$

$$K_f(x) = |f'(x)| \left| \frac{x}{f(x)} \right|$$

$$f(a) = r_1$$

$$f'(a) = \frac{dr_1}{da}$$

$$K_f(a) = \left|\frac{dr_1}{da}\right| \left|\frac{a}{r_1}\right| = \left|\frac{-r_1^2}{a(r_1-r_2)}\right| \left|\frac{a}{r_1}\right| = \left|\frac{r_1}{r_1-r_2}\right|$$

Condition number is large when the difference between the two roots is small compared to r_1.

Note that $K_f(a)$ the condition number didn't depend on the computer or the program used to compute the answer but is a mathematical property of f and how sensitive it is with respect to errors in the input.

In the presence of poor conditioning for a problem $f(x)$, even just the act of rounding the data to floating point may introduce a large change in the result. It's not realistic, then, to expect any algorithm $\tilde{f}$ to have a small error in the sense $\tilde{f}(x) \approx f(x)$. There is another way to characterize the error, though, that can be a useful alternative measurement. Instead of asking, "Did you get nearly the right answer?", we ask, "Did you answer nearly the right question?"

Definition 1.4.1 (Backward error)

Let $\tilde{f}$ be an algorithm for the problem f. Let $y = f(x)$ be an exact result and $\tilde{y} = \tilde{f}(x)$ be its approximation by the algorithm. If there is a value $\tilde{x}$ such that $f(\tilde{x}) = \tilde{y}$, then the relative **backward error** in $\tilde{y}$ is

$$\frac{|\tilde{x} - x|}{|x|}. \tag{1.4.3}$$

The absolute backward error is $|\tilde{x} - x|$.

$y = f(x)$ exact answer $\qquad \tilde{y} = \tilde{f}(x)$ on computer.

What $\tilde{x}$ gives $\qquad \tilde{y} = f(\tilde{x})$

correct function but what input gives the same result as the computer approx.