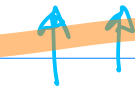


$$f(x) = x + 1$$



cancellation
for x close
to -1

$$\tilde{f}(x) = fl(fl(x) + 1)$$

$$fl(x) = x + \epsilon x \quad \text{where } |\epsilon| \leq \frac{1}{2} \epsilon_{mach}$$

Simplifying assumption that adding 1
can be done exactly..

$$fl(x) + 1 \in \mathbb{F}$$

Thus no generated error from the $\oplus$.

$$\begin{aligned} \text{relative error} &= \frac{|\tilde{f}(x) - f(x)|}{|f(x)|} = \frac{|fl(x) + 1 - (x + 1)|}{|x + 1|} = \\ &= \frac{|x + \epsilon x + 1 - x - 1|}{|x + 1|} = \frac{\epsilon |x|}{|x + 1|} \end{aligned}$$

Note: if x is close to -1 then $x + 1$ is close to 0
and the relative error is really big!

Loss of precision due to subtracting two
nearly equal numbers..

Condition number: $f: \mathbb{R} \rightarrow \mathbb{R} \quad \tilde{x} = fl(x)$

$$\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \quad \left| \frac{\tilde{x} - x}{x} \right|$$

relative error in the
output of f with
respect to relative
error in the input

What happens in the limit of better and better approximations
of x by $\tilde{x}$.

Suppose $\tilde{x} = x + \epsilon x$

where $0 < \epsilon < \frac{1}{2} \epsilon_{\text{mach}}$.

$$\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \bigg/ \left| \frac{\tilde{x} - x}{x} \right| = \left| \frac{f(x + \epsilon x) - f(x)}{f(x)} \right| \bigg/ \left| \frac{x + \epsilon x - x}{x} \right|$$

So denominator is not zero

small change

$$= \left| \frac{f(x + \epsilon x) - f(x)}{\epsilon f(x)} \right| = \left| \frac{f(x + \epsilon x) - f(x)}{\epsilon x} \right| \left| \frac{x}{f(x)} \right|$$

→

$$\left| f'(x) \right| \left| \frac{x}{f(x)} \right| \quad \text{as } \epsilon \rightarrow 0$$

$k_f(x)$ relative condition number of f at x

Since

$$\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \bigg/ \left| \frac{\tilde{x} - x}{x} \right| \rightarrow k_f(x) \quad \text{as } \epsilon \rightarrow 0$$

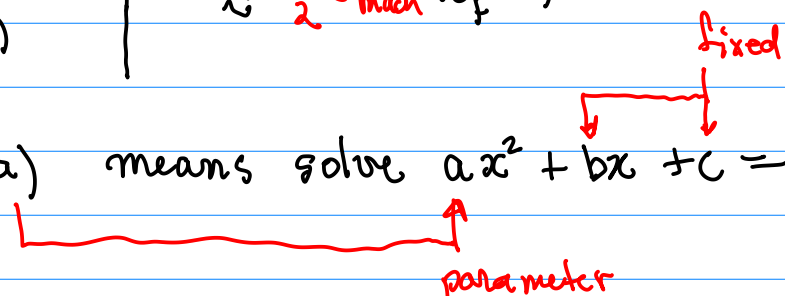
then if $\epsilon \approx 0$ then

$$\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \bigg/ \left| \frac{\tilde{x} - x}{x} \right| \approx k_f(x)$$

biggest is $\frac{1}{2} \epsilon_{\text{mach}}$

Therefore ...

$$\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \leq \frac{1}{2} \varepsilon_{\text{mach}} K_f(x)$$

Suppose $f(a)$ means solve $ax^2 + bx + c = 0$ for x


$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

To make specific, $f(a) = r_1$,

Condition of the root r_1 on the parameter a .

Implicit differentiation,

$$\frac{d}{da} (a r_1^2 + b r_1 + c) = \frac{d}{da} 0$$

$$\left(\frac{da}{da} \right) r_1^2 + a \frac{dr_1^2}{da} + b \frac{dr_1}{da} = 0$$

$$r_1^2 + 2a r_1 \frac{dr_1}{da} + b \frac{dr_1}{da} = 0$$

$$\frac{dr_1}{da} = \frac{-r_1^2}{2a r_1 + b} = \frac{-r_1^2}{a(r_1 - r_2)}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$2a r_1 + b = \sqrt{b^2 - 4ac} = a(r_1 - r_2)$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$2a r_1 + b = \sqrt{b^2 - 4ac}$$

$$r_1 - r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{2\sqrt{b^2 - 4ac}}{2a}$$

$$a(r_1 - r_2) = \sqrt{b^2 - 4ac}$$

$$K_f(x) = \left| f'(x) \right| \left| \frac{x}{f(x)} \right|$$

$$f(a) = r_1$$

$$f'(a) = \frac{dr_1}{da}$$

$$K_f(a) = \left| \frac{dr_1}{da} \right| \left| \frac{a}{r_1} \right| = \left| \frac{-r_1^2}{a(r_1 - r_2)} \right| \left| \frac{a}{r_1} \right| = \left| \frac{r_1}{r_1 - r_2} \right|$$

Condition number is large when the difference between the two roots is small compared to r_1 .

Note that $K_f(a)$ the condition number didn't depend on the computer or the program used to compute the answer but is a mathematical property of f and how sensitive it is with respect to errors in the input,

In the presence of poor conditioning for a problem $f(x)$, even just the act of rounding the data to floating point may introduce a large change in the result. It's not realistic, then, to expect any algorithm $\tilde{f}$ to have a small error in the sense $\tilde{f}(x) \approx f(x)$. There is another way to characterize the error, though, that can be a useful alternative measurement. Instead of asking, "Did you get nearly the right answer?", we ask, "Did you answer nearly the right question?"

Definition 1.4.1 (Backward error)

Let $\tilde{f}$ be an algorithm for the problem f . Let $y = f(x)$ be an exact result and $\tilde{y} = \tilde{f}(x)$ be its approximation by the algorithm. If there is a value $\tilde{x}$ such that $f(\tilde{x}) = \tilde{y}$, then the relative **backward error** in $\tilde{y}$ is

$$\frac{|\tilde{x} - x|}{|x|}. \quad (1.4.3)$$

The absolute backward error is $|\tilde{x} - x|$.

$y = f(x)$ exact answer

What $\tilde{x}$ gives

$\tilde{y} = \tilde{f}(x)$ on computer.

//
 $f(\tilde{x})$

↗
correct function but what
input gives the same
result as the computer
approx.