Fixed point iteration: Trying to find x such that $f(x)=0$.

Idea was set $g(x)=x+cf(x)$ and then iterate

$$x_{n+1}=g(x_n)$$

to obtain a sequence that (hopefully) converges to a root of f.

Let r be the root so $f(r)=0$ and x_0 be an initial approximation of that root. Then we form the sequence $x_{n+1}=g(x_n)$ and ask does it converge to r?

$\varepsilon_n = x_n - r$ then $x_n \to r$ means $\varepsilon_n \to 0$.

from last time

Error: $\varepsilon_n = x_n - \sqrt{2}$. Then by the mean value theorem

$$\varepsilon_{n+1} = x_{n+1} - \sqrt{2} = g(x_n) - g(\sqrt{2}) = g'(\xi_n)(x_n - \sqrt{2}) = g'(\xi_n)\,\varepsilon_n$$

Then $\varepsilon_{n+1} = g'(\xi_n)\,\varepsilon_n$ for some ξ_n between x_n and r.

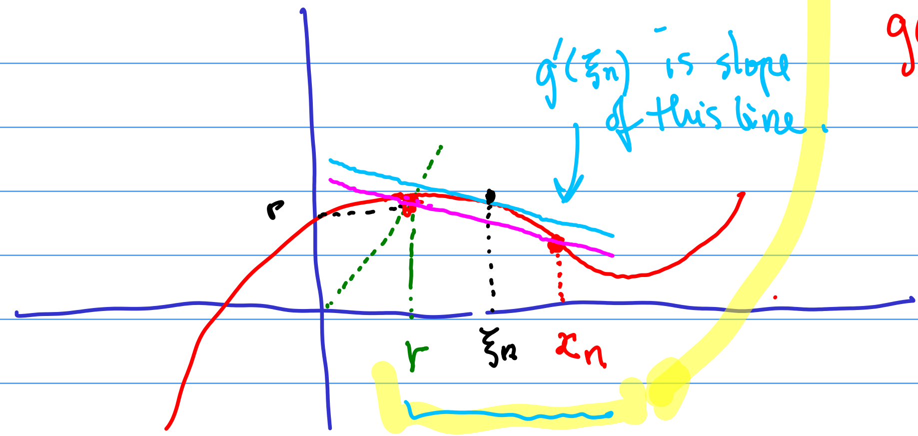
$$|\varepsilon_{n+1}| = |g'(\xi_n)|\,|\varepsilon_n|$$

Note, we need $|g'(\xi_n)|<1$ for the errors to get smaller

Review of mean-value theorem

If $f(r)=0$ then $g(r)=r$

$g(x) = x + c f(x)$



Then $\varepsilon_{n+1} = g'(\xi_n)\,\varepsilon_n$ for some ξ_n between x_n and r.

$$\boxed{|\varepsilon_{n+1}| = |g'(\xi_n)|\,|\varepsilon_n|}$$

need $|g'(\xi_n)| < 1$ for the errors to get smaller

$g(x) = x + c f(x)$ so $\underbrace{g'(x) = 1 + c f'(x)}$

want this close to zero, so the 1 and $cf'(x)$ need to cancel.

Assuming we have $x_n \to r$
then also $\xi_n \to r$ as $n \to \infty$.

$$g'(\xi_n) = 1 + c f'(\xi) \to \underbrace{1 + c f'(r)}$$

smallest this could be is zero.

Then $1 + cf'(r) = 0$ implies $c = \dfrac{-1}{f'(r)}$

Choose $\delta > 0$ so that $|g'(x)| \leq \gamma < 1$ for all $x \in [r-\delta, r+\delta]$ for some γ

Since $g'(r) = 0$ and assuming g' is continuous, then we know there is a $\delta > 0$ so this happens.

Now choose $x_0 \in [r-\delta, r+\delta]$. Then $|\varepsilon_0| = |x_0 - r| \leq \delta$

Then $|\varepsilon_1| = |g'(\xi_0)||\varepsilon_0| < \gamma\delta$ so $|\varepsilon_1| < \gamma\delta$

$|\varepsilon_2| = |g'(\xi_1)||\varepsilon_1| < \gamma|\varepsilon_1| < \gamma^2\delta$

Similarly $|\varepsilon_3| < \gamma^3\delta$, $|\varepsilon_4| < \gamma^4\delta$ and so on...

It follows that $\varepsilon_n \to 0$ and so $x_n \to r$.

One problem: The optimal value for c relies on knowing the root r and that's what we try to find.

Then $1 + cf'(r) = 0$ implies $c = \dfrac{-1}{f'(r)}$ don't know r.

Newton's method: Assume x_0 is close enough to the root that taking $c = \dfrac{-1}{f'(x_0)}$ also gives $|g'(\xi_0)| \leq \gamma < 1$.

Then the iteration $x_{n+1} = x_n + cf(x_n)$ converges to r as $n \to \infty$. from what we just did...

Note, we can do better than this:

$$x_{n+1} = x_n - \frac{1}{f'(x_0)} f(x_n)$$

How?, well $x_n \to r$ means we have a better approximation of c at each iteration

Newton's method..

$$\boxed{x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n)}$$

best approximation of r so far used for $c = \frac{-1}{f'(r)}$

$$\varepsilon_n = x_n - r \quad \longleftrightarrow \quad r - x_n = -\varepsilon_n$$

Then

$$\varepsilon_{n+1} = x_{n+1} - r = x_n - \frac{1}{f'(x_n)} f(x_n) - r$$

$$= \underbrace{x_n - r}_{\varepsilon_n} - \frac{1}{f'(x_n)} f(x_n) = \varepsilon_n - \boxed{\frac{f(x_n)}{f'(x_n)}}$$

estimate this. using Taylor theorem

Taylors theorem

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(\eta)$$

where η is between a and b

Let $b = r$ $a = x_n$

divide to get the quotient

$$0 = f(r) = f(x_n) + (r - x_n) f'(x_n) + \frac{(r-x_n)^2}{2} f''(\eta_n)$$

where η_n is between r and x_n.

$$0 = \frac{f(x_n)}{f'(x_n)} - \varepsilon_n + \frac{\varepsilon_n^2}{2} \frac{f''(\eta_n)}{f'(x_n)}$$

Thus

$$\frac{f(x_n)}{f'(x_n)} = \varepsilon_n - \frac{\varepsilon_n^2}{2} \frac{f''(\eta_n)}{f'(x_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x_n)}{f'(x_n)} = \cancel{\varepsilon_n} - \left(\cancel{\varepsilon_n} - \frac{\varepsilon_n^2}{2} \frac{f''(\eta_n)}{f'(x_n)}\right)$$

Therefore

$$\boxed{|\varepsilon_{n+1}| = \frac{\varepsilon_n^2}{2} \left|\frac{f''(\eta_n)}{f'(x_n)}\right|}$$

Compare

$$\boxed{|\varepsilon_{n+1}| = |g'(\xi_n)||\varepsilon_n|}$$

the next error is proportionately smaller than the square of the previous..

the next error is proportionately smaller than previous.

linear converging

This is called quadratic convergence