# Polynomial interpolation theorem

$$(f(x_0), x_0), \ (f(x_1), x_1) \dots (f(x_n), x_n).$$

Let $g(y) = f^{-1}(y)$ $\boxed{\text{near}}$ the root we're interested in

assumtion near the root the function actually has an inverse...

$$y_0 = f(x_0) \qquad y_1 = f(x_1), \dots, y_n = f(x_n)$$

The points to interpolate are

$p(y_i) = g(y_i)$ for $i = 0, 1, \dots, n$

$$(y_0, g(y_0)), \ (y_1, g(y_1)) \ \dots \ (y_n, g(y_n))$$

Interpolate with a polynomial $p(y)$. Question how good is

$$p(y) \approx g(y).$$

$q(y_i) = 0$ by design...

Let $q(y) = \underbrace{(y-y_0)}_{1}\underbrace{(y-y_1)}_{2}\cdots\underbrace{(y-y_n)}_{n+1}$ polynomial of degree $n+1$

$$\frac{d^{n+1}q(y)}{dy^{n+1}} = \frac{d^{n+1}}{dy^{n+1}}\left(y^{n+1} + \text{lower order terms}\right) = (n+1)!$$

Let $E$ be the error in the approximation:

$$E(y) = p(y) - g(y)$$

note $y^* \neq y_i$ for all $i = 0, \dots, n$.

Let $y^*$ be a point of interest where we will estimate the error

$$F(y) = E(y) + \alpha q(y)$$

constant to be determined

Properties of $F(y)$

recall

$p(y_i) = g(y_i)$
$q(y_i) = 0$

$$F(y_i) = E(y_i) + \alpha q(y_i) = \cancel{p(y_i)} - \cancel{g(y_i)} + \alpha q(y_i) = 0$$

so $F(y)$ has a root at every $y_i$ for $i = 0, \dots, n$
that means $n+1$ roots...

Idea: adjust $\alpha$ the constant to make one more root at $y^*$

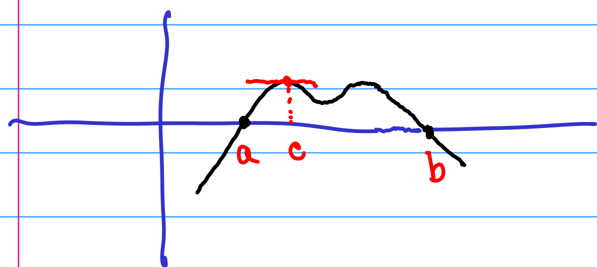
$$F(y^*) = E(y^*) + \alpha q(y^*) = 0 \qquad \alpha = \frac{-E(y^*)}{q(y^*)}$$

Therefore

$$F(y) = E(y) - \frac{E(y^*)}{q(y^*)} q(y)$$

has $n+2$ roots. They are at $y^*, y_0, y_1, \ldots, y_n$.

Rolle's Theorem. If $f(a)=0$ and $f(b)=0$ and $f$ is differentiable on $(a,b)$ and continuous on $[a,b]$. Then there is a point $\boxed{c \in (a,b)}$ such that $f'(c)=0$.




Proof: somewhere between a and b the function f has a minimum or a maximum at that extreme value $f'(c)=0$ since it's not at an end point.

Between $n+2$ roots are $n+1$ intervals in each interval is a place where the derivative is zero.

$F(y)$ has $n+2$ roots

$F'(y)$ has $n+1$ roots

$F''(y)$ has $n$ roots

⋮

$F^{(n)}(y)$ has 2 roots

$F^{(n+1)}(y)$ has 1 root.

between $\min(y^*, y_0, y_1, \ldots, y_n)$ and $\max(y^*, y_0, y_1, \ldots, y_n)$

recall

$$\frac{d^{n+1}q(y)}{dy^{n+1}} = (n+1)!,$$

$$E(y) = p(y) - g(y)$$

Let $c$ be the root so that $F^{(n+1)}(c) = 0$.

const

$$F^{(n+1)}(y) = E^{n+1}(y) - \boxed{\frac{E(y^*)}{q(y^*)}} q^{n+1}(y) = p^{(n+1)}(y) - g^{(n+1)}(y) - \frac{E(y^*)}{q(y^*)}(n+1)!$$

interpolating polynomial through $n+1$ points

general func.

$(y_0, g(y_0)),\ (y_1, g(y_1))\ \cdots\ (y_n, g(y_n))$ $n+1$ points degree $n$ polynomial passes through them

Interpolate with a polynomial $p(y)$. What is the degree of $p$? $n$

$$p^{(n+1)}(y) = 0$$

differentiated $n+1$ times a polynomial of degree $n$

Thus

$$F^{(n+1)}(y) = -g^{(n+1)}(y) - \frac{E(y^*)}{q(y^*)}(n+1)!$$

$$F^{(n+1)}(c) = 0 = -g^{(n+1)}(c) - \frac{E(y^*)}{q(y^*)}(n+1)!$$

recall $E(y) = p(y) - g(y)$

Therefore $E(y^*) = -q(y^*)\dfrac{g^{(n+1)}(c)}{(n+1)!}$

$$p(y^*) - g(y^*) = -q(y^*)\frac{g^{(n+1)}(c)}{(n+1)!}$$

Consequently.

$$\boxed{g(y^*) = \underbrace{p(y^*)}_{\text{approximation}} + \underbrace{q(y^*)\frac{g^{(n+1)}(c)}{(n+1)!}}_{\text{remainder term}}}$$

This is like Taylors theorem

note that the equality also holds even when $y^* = y_i$ for some $i$.

Therefore $y^*$ can be arbitrary and we can write

$$g(y) = p(y) + q(y)\frac{g^{(n+1)}(c)}{(n+1)!} \quad \text{for any } y.$$

where $c$ is some value between $\min(y, y_0, y_1, \ldots, y_n)$ and $\max(y, y_0, y_1, \ldots, y_n)$.

# Interpolating polynomial Theorem

Suppose $g$ is a function with $n+1$ derivatives and $p$ is the interpolating polynomial of degree $n$ through $(y_i, g(y_i))$ for $i = 0, 1, \dots, n$. Then

$$g(y) = p(y) + (y-y_0)(y-y_1)\cdots(y-y_n)\frac{g^{(n+1)}(c)}{(n+1)!}$$

where $c$ is some value between $\min(y, y_0, y_1, \dots, y_n)$ and $\max(y, y_0, y_1, \dots, y_n)$.