$$|x+iy| = \sqrt{x^2+y^2}$$

intermediate part of the calc. may overflow...

rearrange the terms to avoid the overflow:

Case $|x| > |y|$

more likely to overflow

$$|x+iy| = \sqrt{x^2+y^2} = |x| \sqrt{1+(y/x)^2}$$

since $\left|\frac{y}{x}\right| < 1$ then the intermediate computation can't overflow...

Case $|y| > |x|$

more likely to overflow

$$|x+iy| = \sqrt{x^2+y^2} = |y| \sqrt{\left(\frac{x}{y}\right)^2 + 1}$$

since $\left|\frac{x}{y}\right| < 1$ then the intermediate computation can't overflow...

```
julia> f(x,y)=sqrt(x^2+y^2)
f (generic function with 1 method)

julia> x=3e200
3.0e200

julia> y=4e200
4.0e200
```

$$\sqrt{(3\times10^{200})^2+(4\times10^{200})^2} = 10^{200}\sqrt{3^2+4^2} = 10^{200}\sqrt{25}$$

$$= 5\times10^{200}$$

```
julia> f(x,y)
Inf
```

```
julia> function f2(x,y)
           if abs(x)>abs(y)
               return abs(x)*sqrt(1+(y/x)^2)
           else
               return abs(y)*sqrt((x/y)^2+1)
           end
       end
f2 (generic function with 1 method)

julia> f2(x,y)
4.9999999999999995e200
```

```
julia> f2(1.0,1.0)
1.4142135623730951

julia> sqrt(2)
1.4142135623730951

julia> f2(1.0,0.0)
1.0

julia> f2(0.0,1.0)
1.0

julia> f2(0.0,0.0)
NaN
```

bug...

```
julia> function f2(x,y)
           if abs(x)>abs(y)
               return abs(x)*sqrt(1+(y/x)^2)
           elseif abs(y)>abs(x)
               return abs(y)*sqrt((x/y)^2+1)
           else
               return 0.0
           end
       end
f2 (generic function with 1 method)

julia> f2(0.0,0.0)
0.0
```

did this actually fix the bug?

To fix overflow rescale the numbers used in the intermediate computations...

$$f(x) = \sqrt{x}$$

$$f(x^*) = \sqrt{x^*} = \sqrt{x(1+\delta)} = \sqrt{x}\,\underbrace{\sqrt{1+\delta}}_{\text{expand this}}$$

$$= \sqrt{x}\left(1 + \frac{1}{2}\delta - \frac{1}{4}\delta^2 + \cdots\right)$$

$$\approx \sqrt{x}\left(1 + \frac{1}{2}\delta\right)$$

$\left.\frac{1}{2}x^{-1/2}\right|_{x=1} = \frac{1}{2}$

$\left.\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^{-3/2}\right|_{x=1} = \frac{-1}{4}$

The relative error is now $\frac{1}{2}\delta$ in the output.

$$\text{relative error in the output} = \frac{f(x^*) - f(x)}{f(x)} \approx \frac{\cancel{\sqrt{x}}\left(\cancel{1} + \frac{1}{2}\delta\right) - \cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = \frac{1}{2}\delta$$

$$\text{relative error in the input} = \frac{x^* - x}{x} = \frac{\cancel{x}(\cancel{1}+\delta) - \cancel{x}}{\cancel{x}} = \delta$$

$$\text{ratio} = \underbrace{\frac{\text{relative error in the output}}{\text{relative error in the input}}}_{\text{Called relative condition number}} = \frac{\frac{f(x^*) - f(x)}{f(x)}}{\frac{x^* - x}{x}} \approx \frac{\frac{1}{2}\cancel{\delta}}{\cancel{\delta}} = \frac{1}{2}$$

$$K_f(x) = \lim_{\delta \to 0} \frac{\frac{f(x^*) - f(x)}{f(x)}}{\frac{x^* - x}{x}} = \frac{1}{2}$$

assume δ is a small error and then take $\delta \to 0$ to simplify things.

not an approximation because $\delta \to 0$.

Suppose y^* is an approximation of $\sqrt{x}$

want to find the error $\frac{y^*-y}{y}$ where $y=\sqrt{x}$

rel. error in y^*

$(y^*)^2 \approx x$

residual error $= (y^*)^2 - x = x^* - x$

know what x is that the input but don't know y

Let $x^* = (y^*)^2$

relative residual error $= \frac{x^*-x}{x}$

Let $f(x)=\sqrt{x}$ and $g(y)=f^{-1}(y)=y^*$

$$\frac{\frac{f(x^*)-f(x)}{f(x)}}{\frac{x^*-x}{x}} \approx K_f(x) = \frac{1}{2}$$

$$\frac{y^*-y}{y} = \frac{f(x^*)-f(x)}{f(x)} \approx K_f(x)\,\frac{x^*-x}{x}$$

Another example:

Suppose y^* is an approximation of $y=e^x$

thus $x^* = \ln y^* \approx x$ want to know $\frac{y^*-y}{y}$

$$\frac{y^*-y}{y} \approx K_f(x)\,\frac{x^*-x}{x}, \qquad f(x)=e^x$$

we need a general formula for $K_f(x)$

next time...