Suppose y^* is an approximation of $y = e^x$

thus $x^* = \ln y^* \approx x$ want to know $\dfrac{y^*-y}{y}$

$$\frac{y^*-y}{y} \approx K_f(x)\,\frac{x^*-x}{x}, \qquad f(x) = e^x$$

$$\delta = \frac{x^*-x}{x} \qquad x\delta = x^*-x \qquad x^* = x(1+\delta)$$

relative error in output

$$K_f(x) = \lim_{\delta\to 0} \frac{f(x^*)-f(x)}{f(x)} \Big/ \frac{x^*-x}{x}$$

relative error in input

definition of derivative

$$= \lim_{\delta\to 0} \frac{f(x(1+\delta))-f(x)}{f(x)}\cdot\frac{1}{\delta} = \frac{x}{f(x)} \lim_{\delta\to 0} \boxed{\frac{f(x+\delta x)-f(x)}{\delta x}}$$

$$= \frac{x}{f(x)}\, f'(x) = \boxed{\frac{x f'(x)}{f(x)}}$$

$$\frac{y^*-y}{y} \approx \frac{x f'(x)}{f(x)}\,\frac{x^*-x}{x}$$

<u>Example</u>: Suppose that y^* was obtained by the approx.

$$e^x \approx \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}_{\text{first four terms of Taylor series...}}$$

Let's use this to approximate $e = e^1$. So $x = 1$

$$y^* = 1+1+\frac{1}{2}+\frac{1}{6} = \frac{6+6+3+1}{6} = \frac{16}{6} = \frac{8}{3}$$

What is the error in this approximation?

```
julia> ystar=1+1+1/2+1/6
2.6666666666666665

julia> (ystar-exp(1))/exp(1)
-0.018988156876153812
```

Direct computation:

$$\frac{y^*-e}{e} = -0.018988156876153812$$

Backwards error analysis: $x^* = \ln y^*$

pretend we alread know how to compute ln.

$$\frac{y^*-y}{y} \approx \frac{x f'(x)}{f(x)} \frac{x^*-x}{x}$$

$f(x) = e^x$ $\qquad K_f(x) = \frac{x f'(x)}{f(x)} = \frac{x \cancel{e^x}}{\cancel{e^x}} = x$

know $\qquad$ know

$$\frac{y^*-y}{y} \approx K_f(x) \frac{x^*-x}{x} = \cancel{x} \frac{x^*-x}{\cancel{x}} = x^*-x = \ln y^* - 1$$

```
julia> log(ystar)-1
-0.019170746988273812
```

$= -0.019170746988273812$

Linear algebra example...

Idea: $\qquad$ Solve... $Ax = b$ $\qquad$ input A, b

output x

We have an approximation of x called x^*. What is the error?

$\varepsilon = x^* - x$ ← exact solution.

$\delta = \frac{\|x^*-x\|}{\|x\|}$ relative error want to estimate this using backwards error analysis...

In this case $f(b) = A^{-1}b$ then $x \approx f(b)$

If $x = f(b)$ then $Ax = b$

But what about x^*? $Ax^* = b^*$ ← definition of b^*

$f^{-1}(x) = Ax$ $\qquad b^* = f^{-1}(x^*)$

estimate the error using backwards

$$\frac{\|x^* - x\|}{\|x\|} \approx K_f(x) \frac{\|b^* - b\|}{\|b\|}$$

find this next time...

Back to Lab 1

$$\text{xplus}(a,b,c) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\text{xplusnew}(a,b,c) = \frac{2c}{-b - \sqrt{b^2 - 4ac}}$$

If $b > 0$ and $b^2 \gg |4ac|$ then $\sqrt{b^2 - 4ac} \approx b$

```
julia> xplus(1,10,-1)
0.09901951359278449

julia> xplusnew(1,10,-1)
0.09901951359278484
```

```
julia> xplusnew(1,100,-1)
0.009999000199950016

julia> xplus(1,100,-1)
0.009999000199947261
```

```
julia> xplusnew(1,1000,-1)
0.000999999000002

julia> xplus(1,1000,-1)
0.0009999989999869285
```

```
julia> xplus(Float32(1),Float32(1000),Float32(-1))
0.0010070801f0

julia> xplusnew(Float32(1),Float32(1000),Float32(-1))
0.000999999f0
```

```
julia> xplus(Float16(1),Float16(100),Float16(-1))
Float16(0.0)

julia> xplusnew(Float16(1),Float16(100),Float16(-1))
Float16(0.01)
```

is correctly rounded...

```
julia> xplusnew(1,100,-1)
0.009999000199950016

julia> xplus(1,100,-1)
0.009999000199947261
```

Computation of these

$$\frac{e^z - 1}{z} \qquad \frac{\cos z - 1}{z^2}$$

where z is small.