Math 467/667: Programming Project 1

Your work should be presented in the form of a typed report using clear and properly punctuated English. Pencil and paper calculations may be typed or hand written. Where appropriate include full program listings and output. Please work individually on this project.

1. This question considers approximation of the area

$$A = \int_0^{\pi/2} f(z)dz \qquad \text{where} \qquad f(z) = \frac{1}{\sqrt{1+\tan z}}$$

by various numerical and algebraic techniques.

(i) Use a computer algebra system (or pencil and paper if you prefer) to verify the exact value of A is given by

$$A = \frac{\alpha}{4}\left\{\pi + \frac{3-\alpha^2}{\sqrt{2}}\ln\left(\frac{\alpha-\sqrt{2}}{\alpha+\sqrt{2}}\right) + 4\arctan(\alpha^2 - \alpha\sqrt{2})\right\}$$

where $\alpha = \sqrt{1+\sqrt{2}}$ and then show that $A \approx 1.060233292270744$.

(ii) Modify the program written in class or write your own to approximate A using the composite Gauss quadrature rule of order $\mathcal{O}(h^{10})$ where $h = (b-a)/m$. Compute

$$A_m = \texttt{comp}(f, 0, \pi/2, m) \qquad \text{and} \qquad E_m = A_m - E$$

for $m = 2^\ell$ and $\ell = 1, 2, \ldots, 12$. The output should look like

```
l    2^l                   Am                      Em
1      2  1.060677017116208e+00  4.437248454676190e-04
2      4  1.060389233493366e+00  1.559412226259660e-04
3      8  1.060288268508819e+00  5.497623807904084e-05
```

Note values for $l > 3$ have been omitted in the above table.

(iii) Plot $\log E_m$ versus $\log h$ for the output obtained in the previous question. Do the points lie on a straight line? What is the slope of the line? What does the slope of the line indicate about the numerically observed order of convergence for this calculation? Is the numerically observed order of convergence consistent with what was expected theoretically? Explain.

(iv) Make the change of variables $y = \tan z$ to transform the integral appearing in question (i) to the form

$$\int_0^\infty g(y)dy.$$

Write down an explicit formula for $g(y)$.

(v) Show that the further change of variables $x = 2y/(1+y) - 1$ transforms the integral in question (iv) into

$$\int_{-1}^{1} h(x)\sqrt{1-x}\,dx \qquad \text{where} \qquad h(x) = \frac{2^{-1/2}}{1+x^2}.$$

(vi) Define the weighted inner product and norm as

$$(f,g) = \int_{-1}^{1} f(x)g(x)\sqrt{1-x}\,dx \qquad \text{and} \qquad \|f\| = \sqrt{(f,f)}.$$

Use a computer algebra system (or pencil and paper if you prefer) to find the orthonormal polynomials p_n with respect to this inner product for $n = 0, 1, \ldots, 8$.

(vii) Find the eight roots x_k of $p_8(x)$ and the corresponding weights w_k such that

$$\int_{-1}^{1} x^j \sqrt{1-x}\,dx = \sum_{k=0}^{7} w_k x_k^j \qquad \text{for} \qquad j = 0, 1, \ldots, 15.$$

For reference the roots and weights you find should be consistent with

k	x_k	w_k
0	-0.9624795445887677	0.1340407182534346
1	-0.8075678953806377	0.2835409515409297
2	-0.5496419355080006	0.3727176289987073

Note values for $k > 2$ have been omitted from the above table.

(viii) Use the weighted eight-point Gauss quadrature method developed above to approximate the area A. What is the error in this approximation? How does the composite formula used in question (ii) compare in terms of computational effort?

(ix) [Extra Credit and Math/CS 667] Let `quad_w(`f`)` be the weighted eight-point method above and let `quad(`f`)` be the standard five-point Gauss rule. Show that

$$\int_{b-h}^{b} \phi(t)\sqrt{b-t}\,dt = \frac{h^{3/2}}{2^{3/2}} \int_{-1}^{1} \phi\Big(\frac{xh}{2} + b - \frac{h}{2}\Big)\sqrt{1-x}\,dx.$$

Setting $h = (b-a)/m$ leads to the hybrid composite quadrature formula

$$\int_a^b f(x)dx \approx \texttt{hybrid}(f,a,b,m) = \sum_{j=0}^{m-2} \texttt{quad}(g_j) + \texttt{quad\_w}(\psi)$$

where $f(x) = \phi(x)\sqrt{b-x}$,

$$g_j(x) = \frac{h}{2} f\Big(\frac{xh}{2} + a + hj + \frac{h}{2}\Big) \quad \text{and} \quad \psi(x) = \frac{h^{3/2}}{2^{3/2}}\phi\Big(\frac{xh}{2} + b - \frac{h}{2}\Big).$$

Numerically determine the order of convergence of this method by approximating

$$\frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{\sqrt{1-x}}{1+x^2}\,dx$$

following a similar procedure as in questions (ii) and (iii). Is there a way to fix this method so it converges faster?