Recall: Solving $y' = f(t,y)$ such that $y(t_0) = y_0$

Started with $O(h^2)$ for how much error is generated at each step...

$$\|e_{n+1}\| \leq (1+h\lambda)\|e_n\| + O(h^2)$$

since $h = \frac{T-t_0}{N}$

then $N = \frac{T-t_0}{h}$

and after $N$ intervals one of the $h$'s is gone...

Thus

$$\|e_{n+1}\| \leq (1+h\lambda)\|e_n\| + ch^2$$

Use induction...  $e_0 = y_0 - y(t_0) \approx 0$ up to rounding errors..

$$\|e_1\| \leq (1+h\lambda)\|e_0\| + ch^2$$

$$\|e_2\| \leq (1+h\lambda)\|e_1\| + ch^2$$

$$\leq (1+h\lambda)\{(1+h\lambda)\|e_0\| + ch^2\} + ch^2$$

$$= (1+h\lambda)^2\|e_0\| + \{1 + (1+h\lambda)\}ch^2$$

$$\|e_3\| \leq (1+h\lambda)\|e_2\| + ch^2$$

$$\leq (1+h\lambda)\{(1+h\lambda)^2\|e_0\| + \{1+(1+h\lambda)\}ch^2\} + ch^2$$

$$= (1+h\lambda)^3\|e_0\| + (\underbrace{1 + (1+h\lambda) + (1+h\lambda)^2}_{\text{geometric sum}})ch^2$$

one less than 3

In general

$$\|e_n\| \leq (1+h\lambda)^n\|e_0\| + \underbrace{\sum_{k=0}^{n-1}(1+h\lambda)^k}_{\text{simplify this geometric sum...}} ch^2$$

$S_n = \sum_{k=0}^{n-1} \alpha^k$ ← simplify this....

$$\alpha S_n = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^n$$
$$S_n = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$$
$$(\alpha - 1) S_n = \alpha^n - 1$$

Thus $\sum_{k=0}^{n-1} \alpha^k = \frac{\alpha^n - 1}{\alpha - 1}$

Therefore

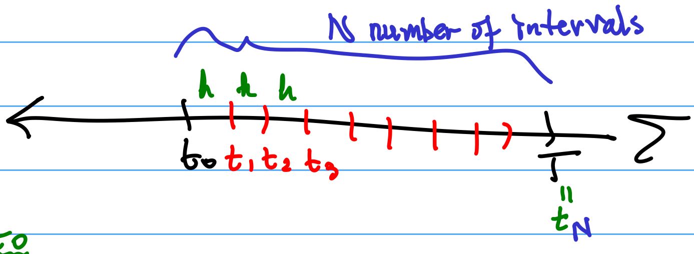
$$\|e_n\| \le (1+h\lambda)^n \|e_0\| + \sum_{k=0}^{n-1} (1+h\lambda)^k ch^2$$
$$= (1+h\lambda)^n \|e_0\| + \frac{(1+h\lambda)^n - 1}{(1+h\lambda) - 1} ch^2$$
$$= (1+h\lambda)^n \|e_0\| + \frac{(1+h\lambda)^n - 1}{\lambda} ch$$

Now I want to work with $(1+h\lambda)^n$ and bound this...

Remember the setup

$$y' = f(t,y) \qquad y(t_0) = y_0$$

Made a grid. Solving on $[t_0, T]$ and




$h = \frac{T - t_0}{N}$

Check that $t_N = T$

$t_N = t_0 + hN = t_0 + \frac{T - t_0}{N} N = T$

$t_n = t_0 + hn$

$hn = t_n - t_0$

$$\|e_n\| \le (1+h\lambda)^n \|e_0\| + \frac{(1+h\lambda)^n - 1}{\lambda} ch$$

$$e^{h\lambda} = 1 + \underbrace{h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{3!}(h\lambda)^3 + \frac{1}{4!}(h\lambda)^4 + \dots}_{\text{all these terms are positive since } h\lambda > 0}$$

Thus,

$$1 + h\lambda \leq e^{h\lambda}$$

$$\|e_n\| \leq (e^{h\lambda})^n \|e_0\| + \frac{ch}{\lambda}\left[(e^{h\lambda})^n - 1\right]$$

$$= e^{hn\lambda} \|e_0\| + \frac{ch}{\lambda}\left[e^{hn\lambda} - 1\right]$$

$$= e^{(t_n - t_0)\lambda} \|e_0\| + \frac{ch}{\lambda}\left[e^{(t_n - t_0)\lambda} - 1\right]$$

Since $t_n \leq T$

Then

$$\|e_n\| \leq \underbrace{e^{(T-t_0)\lambda} \|e_0\| + \frac{ch}{\lambda}\left[e^{(T-t_0)\lambda} - 1\right]}_{\text{doesn't depend on } n.}$$

Recall

The method converges on $[t_0, T]$ if

$$\lim_{N \to \infty} \left( \max \{ \|e_n\| : n = 1, \dots, N \} \right) = 0$$

and we a trying to show Euler's method converges...

this term goes to zero in the limit.

$$\lim_{N \to \infty} \max \left\{ \underbrace{e^{(T-t_0)\lambda} \|e_0\|}_{\text{propagated error}} + \overbrace{\frac{ch}{\lambda} \underbrace{\left[e^{(T-t_0)\lambda} - 1\right]}_{\text{const}}}^{\text{generated error...}} \right.$$

const

generated by the truncation in each step of the integral on the right

$$h = \frac{T - t_0}{N} \to 0 \quad \text{as } N \to \infty$$

$$= e^{(T-t_0)\lambda} \|e_0\| = 0$$

up to the rounding error in the initial condition...

This also tells us the rate of convergence when there is no rounding error...

$$\text{total error} \le \frac{ch}{\lambda}\left[e^{(T-t_0)\lambda} - 1\right] = O(h)$$

This is called a first order method...

Generalize the argument...

$$y' = f(t,y) \qquad y(t_0) = y_0$$

$$t_n = t_0 + hn \qquad h = \frac{T-t_0}{N}$$

Integrate the ODE on both sides from $t_n$ to $t_{n+1}$

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t,y(t))\,dt$$

$y(t_{n+1}) - y(t_n) =$ Some approximation...

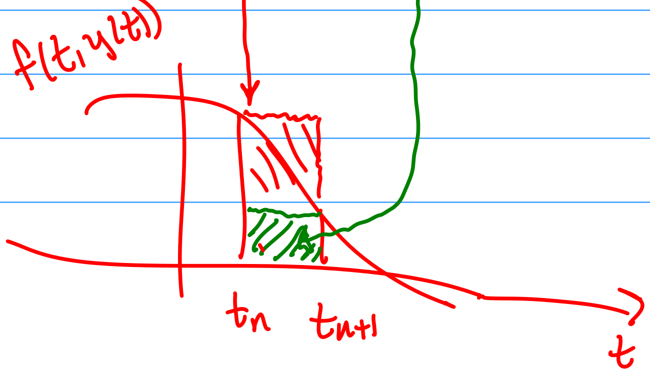
Euler's method

left endpoint rectangle. $f(t_n, y(t_n))\,h$

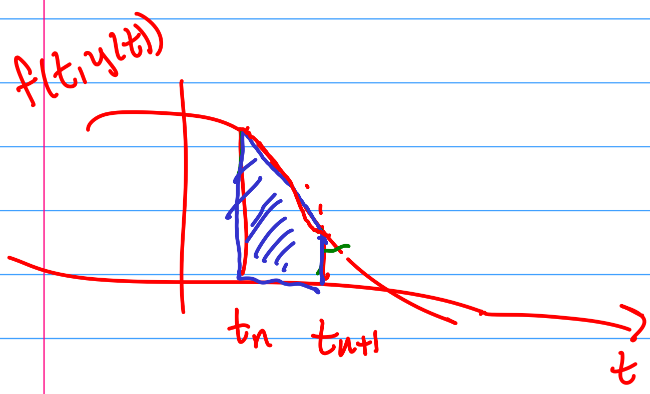
right endpoin rectangle $f(t_{n+1}, y(t_{n+1}))\,h$

Backwards differentiation formula

or: Backwards Euler.



$\boxed{\text{trapezoid rule}}$ $\quad \dfrac{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))}{2} h$




Introduce a parameter $\theta$ and approximate the integral as a weighted average of the left and right endpoints

$\boxed{\text{Theta Method}}$ $\quad \left[\theta f(t_n, y(t_n)) + (1-\theta) f(t_{n+1}, y(t_{n+1}))\right] h$

Two questions:

① Does it converges as $N \to \infty$?

② How fast?

Theoretical assumption of no rounding error..

Trapezoid rule:

$$y(t_{n+1}) - y(t_n) \approx \frac{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))}{2} h$$

Numerical scheme

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h$$

use this equation to solve for $y_{n+1}$ given $y_n$.

Since $y_{n+1}$ appears on the right side in the function $f$, this is called an implicit method.

Solving for $y_{n+1}$ using Newton's method is not so bad because you can use $y_n$ as the initial guess so it only take 1 or 2 iterations to find $y_{n+1}$

Computer lab on practical implicit methods.

Analysis of this:

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h$$

Truncation error $\tau_n$. plug the exact solution into the numerical scheme and compute the residual error.

$$y(t_{n+1}) - y(t_n) = \frac{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))}{2} h + \tau_n$$

Estimate $\tau_n$:

$$\tau_n = y(t_{n+1}) - y(t_n) - \frac{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))}{2} h$$

mult by $h$ here

Since $y' = f(t, y)$ then $f(t_n, y(t_n)) = y'(t_n)$

and $f(t_{n+1}, y(t_{n+1})) = y'(t_{n+1})$

Taylor series expand everything

Set $t := t_n$ for convenience

$$y(t_{n+1}) = y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$$

$$y'(t_{n+1}) = y'(t+h) = y'(t) + h\,y''(t) + O(h^2)$$

In the end everything cancels, check this on thursday...

$$\tau_n = O(h^3).$$