$$\tau_n = y(t_{n+1}) - y(t_n) - \frac{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))}{2} h = O(h^3)$$

Conclusion: Provided the method converges $E_N = O(h^2)$

Show Trapezoid method converges

$$y' = f(t,y) \qquad\qquad y(t_0) = y_0$$

Solving on the interval $[t_0, T]$ using N equally spaced time steps... Grid $t_n = t_0 + h n$ where $h = \frac{T - t_0}{N}$.

Trapezoid method: integrate both sides of the ODE over one time step...

$$\underbrace{\int_{t_n}^{t_{n+1}} y'(t)\,dt}_{\text{fundamental theorem}} = \underbrace{\int_{t_n}^{t_{n+1}} f(t, y(t))\,dt}_{\text{approximate using}}$$

trapezoid rule

$$y(t_{n+1}) - y(t_n) \approx \tfrac{1}{2} h \left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right)$$

$$\tau_n = y(t_{n+1}) - y(t_n) - \tfrac{1}{2} h \left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right) = O(h^3)$$

Trapezoid method

$$y_{n+1} = y_n + \tfrac{1}{2} h \left(f(t_n, y_n) + f(t_n, y_{n+1}) \right) \qquad \text{where } y_n \approx y(t_n)$$

Convergence: Need to show error goes to zero as $N \to \infty$ or $h \to 0$

$$e_{n+1} = y_{n+1} - y(t_{n+1}) = y_n + \tfrac{1}{2} h \left(f(t_n, y_n) + f(t_n, y_{n+1}) \right)$$

$$- \underbrace{\tau_n}_{O(h^3)} - y(t_n) - \tfrac{1}{2} h \left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right)$$

← This is e_n

$$e_{n+1} = e_n + \frac{1}{2}h\big(f(t_n,y_n) - f(t_n,y(t_n))\big) + \frac{1}{2}h\big(f(t_n,y_{n+1}) - f(t_{n+1},y(t_{n+1}))\big) + O(h^3)$$

hypothesis on f

Recall f is assumed to be Lipschitz cont. in the second variable.... Thus

$$|f(t,y) - f(t,z)| \leq \lambda|y-z| \quad \text{for all } y,z.$$

$$|e_{n+1}| \leq |e_n| + \frac{1}{2}h\lambda|y_n - y(t_n)| + \frac{1}{2}h\lambda|y_{n+1} - y(t_{n+1})| + O(h^3)$$

$$|e_{n+1}| \leq |e_n| + \frac{1}{2}h\lambda|e_n| + \frac{1}{2}h\lambda|e_{n+1}| + O(h^3)$$

$$\underbrace{\left(1 - \frac{1}{2}h\lambda\right)}_{\text{has to be positive for this inequality to be useful...}}|e_{n+1}| \leq \left(1 + \frac{1}{2}h\lambda\right)|e_n| + O(h^3)$$

Assume $\frac{1}{2}h\lambda < 1$ or $h < \frac{2}{\lambda}$

means can't take the steps too big...

$$|e_{n+1}| \leq \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)|e_n| + ch^3$$

Recall Euler's method looked similar

$$\|e_{n+1}\| \leq (1+h\lambda)\|e_n\| + O(h^2)$$

In general

$$|e_{n+1}| \leq \alpha|e_n| + \beta \qquad \alpha = \frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} \qquad \beta = ch^3$$

Since $e_0 = y_0 - y(t_0) = 0$ since $y(t_0) = y_0$ be definition (no rounding error)

By induction

$$|e_1| \leq \alpha|e_0| + \beta = \beta$$

$$|e_2| \leq \alpha|e_1| + \beta \leq \alpha(\beta) + \beta = (1+\alpha)\beta$$

$$|e_3| \leq \alpha|e_2| + \beta = \alpha\big((1+\alpha)\beta\big) + \beta = (1+\alpha+\alpha^2)\beta$$

in general

$$|e_n| \le \underbrace{(1+\alpha+\cdots+\alpha^{n-1})}_{\text{geometric sum}}\beta$$

recall from Euler's method

$S_n = \sum_{k=0}^{n-1}\alpha^k$ ← simplify this...

$$\alpha S_n = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^n$$
$$S_n = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$$
$$(\alpha-1)S_n = \alpha^n - 1$$

Thus $\sum_{k=0}^{n-1}\alpha^k = \frac{\alpha^n-1}{\alpha-1}$

Simplify

$$|e_n| \le \frac{\alpha^n-1}{\alpha-1}\beta = \frac{\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1}{\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} - 1}\, ch^3$$

Simplify (using division)

$$\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} = \frac{1-\frac{1}{2}h\lambda + h\lambda}{1-\frac{1}{2}h\lambda} = 1 + \frac{h\lambda}{1-\frac{1}{2}h\lambda}$$

Thus

$$|e_n| \le \frac{\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1}{\frac{h\lambda}{1-\frac{1}{2}h\lambda}}\, ch^{\cancel{3}\,2} = \frac{(1-\frac{1}{2}h\lambda)}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1\right)ch^2$$

Already know $h < \frac{2}{\lambda}$ ...
lets suppose even smaller

$h < \frac{1}{\lambda}$

$$1 + \frac{h\lambda}{1-\frac{1}{2}h\lambda} \le 1 + 2h\lambda \le e^{2h\lambda}$$

$$1-\frac{1}{2}h\lambda > 1 - \frac{1}{2}\frac{1}{\lambda}\lambda = \frac{1}{2}$$

recall from Euler's method

$$e^{h\lambda} = 1 + \underbrace{h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{3!}(h\lambda)^3 + \frac{1}{4!}(h\lambda)^4 + \cdots}_{\text{all these terms are positive since } h\lambda > 0}$$

thus,

$$1 + h\lambda \le e^{h\lambda}$$

Therefore

$$|e_n| \le \frac{1-\cancel{\frac{1}{2}h\lambda}}{\lambda}\left(e^{2h\lambda n} \cancel{-1}\right)ch^2 \le \frac{1}{\lambda}e^{2h\lambda n}ch^2$$

recall the grid: $t_n = t_0 + hn$ $\qquad h = \frac{T - t_0}{N}$

$$hn = t_n - t_0 \leq T - t_0$$

$$|e_n| \leq \frac{1}{\lambda} e^{2(T-t_0)\lambda} ch^2$$ ← doesn't depend on n.

$$E_N = \max\{|e_n| : n = 1, \dots, N\} \leq \frac{1}{\lambda} e^{2(T-t_0)\lambda} ch^2 \to 0$$

as $h \to 0$ or $N \to \infty$.

So the trapezoid method converges.

Moving on to the Next chapter Adams-Bashforth Method...

Solving

$$y' = f(t, y)$$

AB methods are particularly useful when evaluating f is computationally intensive...

Idea integrate over a time step...

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

approximate this integral...

Thus

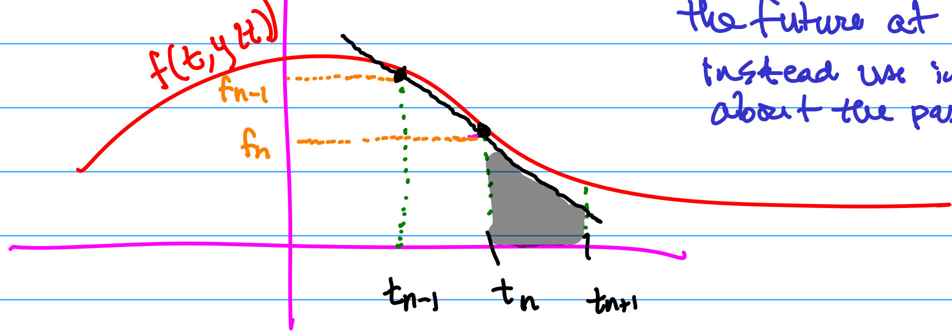
$$y(t_{n+1}) - y(t_n) = \text{approx...}$$

Trapezoid method was nice but implicit...

goal avoid implicit but still have higher order convergence...

Idea

Definition: $f_n = f(t_n, y_n)$



Don't use information about the future at t_{n+1} but instead use information about the past.

Point slope form:

$$\text{slope} = \frac{f_n - f_{n-1}}{t_n - t_{n-1}} = \frac{1}{h}(f_n - f_{n-1})$$

$$\text{point} = (t_n, f_n)$$

$$f - f_n = \frac{1}{h}(f_n - f_{n-1})(t - t_n)$$

$$f(t, y(t)) \approx f_n + \frac{1}{h}(f_n - f_{n-1})(t - t_n)$$

$$\int_{t_n}^{t_{n+1}} f(t, y(t))\,dt = \int_{t_n}^{t_{n+1}} \Big(\underbrace{f_n}_{\text{constant}} + \frac{1}{h}(f_n - f_{n-1})(t - t_n)\Big)\,dt$$

$$= h f_n + \frac{1}{h}(f_n - f_{n-1})\frac{1}{2}(t - t_n)^2 \Big|_{t_n}^{t_{n+1}}$$

$$= h f_n + \frac{1}{h}(f_n - f_{n-1})\frac{1}{2}h^2 = h\left(f_n + \frac{1}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

$$= h\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

Therefore

$$y(t_{n+1}) - y(t_n) \approx h\left(\frac{3}{2}f(t_n, y(t_n)) - \frac{1}{2}f(t_{n-1}, y(t_{n-1}))\right)$$

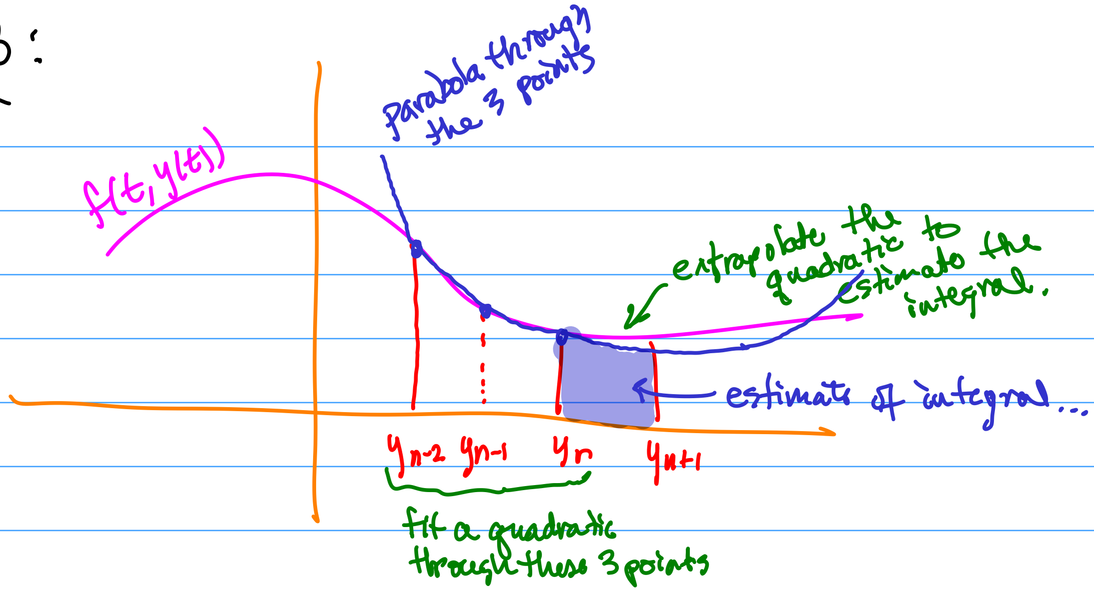
AB2 is then:

$$y_{n+1} = y_n + h\left(\frac{3}{2}f(t_n, y_n) - \frac{1}{2}f(t_{n-1}, y_{n-1})\right)$$

ABk methods:

Remark these methods are based on extrapolating a polynomial fit through k points in the past.

AB3:



parabola through the 3 points
$f(t, y(t))$
extrapolate the quadratic to estimate the integral.
estimate of integral...
y_{n-2} y_{n-1} y_n y_{n+1}
fit a quadratic through these 3 points