notes change from 1.7 to 1.5

HW#1 Exercises 1.1, 1.3, 1.4, 1.5 (lots)

Extra Credit 1.7 ← This is an extension of 1.5.

Try deriving this over the weekend using a computer algebra system

$$y_{n+3} = y_{n+2} + h\left[\tfrac{23}{12} f(t_{n+2}, y_{n+2}) - \tfrac{4}{3} f(t_{n+1}, y_{n+1}) + \tfrac{5}{12} f(t_n, y_n)\right].$$

or the built-in Symbolics and `SymbolicNumericIntegration` packages in Julia...

Chapter 2.2:

2.2 Order and convergence of multistep methods

Idea behind a method was generalize ideas that worked by introducing parameters that can be tuned to make them better.

Tune for

- $\boxed{\text{order of convergence}}$
- stability
- anything else eg. energy conservation properties and other physical constraints...

General multistep method:

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

here a_m's and b_m's are parameters that can be tuned...

This gives y_{n+s} in terms of $\underbrace{y_n, y_{n+1}, \dots, y_{n+s-1}}_{\text{history used to find the next time step...}}$

Note if $b_s \neq 0$ then y_{n+s} also appears on the right side and this is an implicit method..

How to tune the parameters for order... Find the truncation error of this general method as a function of the parameters and then solve for the parameters to make lots of cancellation and find $\tau = O(h^{p+1})$ for some large value of p.

To find the truncation error plug exact solution into

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

and find the residual error

$$\tau = \sum_{m=0}^{s} a_m y(t_{n+m}) - h \sum_{m=0}^{s} b_m f(t_{n+m}, y(t_{n+m}))$$

For simplicity take $n=0$.. There is no loss of generality because what I call the first time step depends on what I call the first time and I can change that however I like.

$$\tau = \sum_{m=0}^{s} a_m y(t_m) - h \sum_{m=0}^{s} b_m f(t_m, y(t_m))$$

Introduce weird notation.

$$\rho(w) = \sum_{m=0}^{s} a_m w^m \quad \text{and} \quad \sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Somehow need a mapping between powers w^m and translations in time $y(t_m) = y(t_0 + mh)$ for this notation to be of use.

How to turn translations in time into powers?

Taylor's Series (provided it converges)

$$y(t_m) = y(t_0+mh) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t_0)}{k!}(mh)^k$$

$$f(t_m, y(t_m)) = y'(t_0+mh) = \sum_{k=0}^{\infty} \frac{y^{(k+1)}(t_0)}{k!}(mh)^k$$

$$\tau = \sum_{m=0}^{s} a_m y(t_m) - h\sum_{m=0}^{s} b_m f(t_m, y(t_m))$$

$$= \sum_{m=0}^{s} a_m \sum_{k=0}^{\infty} \frac{y^{(k)}(t_0)}{k!}(mh)^k - h\sum_{m=0}^{s} b_m \sum_{k=0}^{\infty} \frac{y^{(k+1)}(t_0)}{k!}(mh)^k$$

Now collect powers of h and cancel as many as possible

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!}(mh)^k - \sum_{k=0}^{\infty}\sum_{m=0}^{s} b_m \frac{y^{(k+1)}(t_0)}{k!} m^k h^{k+1}$$

$k=0$

$$= \sum_{m=0}^{s} a_m y(t_0) + \sum_{k=1}^{\infty}\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!}(mh)^k - \underbrace{\sum_{k=0}^{\infty}\sum_{m=0}^{s} b_m \frac{y^{(k+1)}(t_0)}{k!} m^k h^{k+1}}_{\text{rewrite this sum shifting the indices by 1.}}$$

Thus shifting ...

$$\sum_{k=0}^{\infty}\sum_{m=0}^{s} b_m \frac{y^{(k+1)}(t_0)}{k!} m^k h^{k+1} = \sum_{k=1}^{\infty}\sum_{m=0}^{s} b_m \frac{y^{(k)}(t_0)}{(k-1)!} m^{k-1} h^k$$

$$\tau = \left(\sum_{m=0}^{s} a_m\right) y(t_0) + \sum_{k=1}^{\infty}\left(\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!}(mh)^k - \sum_{m=0}^{s} b_m \frac{y^{(k)}(t_0)}{(k-1)!} m^{k-1} h^k\right)$$

needs to be zero for any convergence at all

What do I want? $\tau = O(h^{p+1})$ for $p \geq 1$.

① $\sum_{m=0}^{s} a_m = 0$

Also $\sum_{k=1}^{\infty} \left(\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!} m^k - \sum_{m=0}^{s} b_m \frac{y^{(k)}(t_0)}{(k-1)!} m^{k-1} \right) h^k$

term implies

② $\sum_{m=0}^{s} a_m \frac{1}{k!} m^k - \sum_{m=0}^{s} b_m \frac{1}{(k-1)!} m^{k-1} = 0$ for $k = 1, \ldots, p$

If everything above cancells then $\tau = O(h^{p+1})$ and so the order of the method is $O(h^p)$ provided it converges.

To ensure the resulting method is exactly $O(h^p)$ and not more then

$$\sum_{m=0}^{s} a_m \frac{1}{(p+1)!} m^{p+1} - \sum_{m=0}^{s} b_m \frac{1}{(p)!} m^p \neq 0$$

In terms of ρ and σ what do we have?

$$\rho(w) = \sum_{m=0}^{s} a_m w^m \quad \text{and} \quad \sigma(w) = \sum_{m=0}^{s} b_m w^m$$

① $\sum_{m=0}^{s} a_m = 0 \iff \rho(1) = 0$

Recall

$$\rho(w) = \sum_{m=0}^{s} a_m w^m \quad \text{and} \quad \sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Then

$$\sum_{m=0}^{s} a_m \frac{1}{k!} m^k - \sum_{m=0}^{s} b_m \frac{1}{(k-1)!} m^{k-1} = 0$$

$$\sum_{k=0}^{\infty} \frac{1}{k!}(zm)^k$$

Exponential $w = e^z$ then $w^m = e^{zm} = \sum_{k=0}^{\infty} \frac{1}{k!}(zm)^k$ then $\log w^m = zm$

Hm... it's a bit difficult... there is the theorem...

Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(w) - \sigma(w)\ln w = c(w-1)^{p+1} + \mathcal{O}(|w-1|^{p+2}), \qquad w \to 1. \qquad (2.10)$$

$w = e^z$

$$\rho(w) - \sigma(w)\ln w = \sum_{m=0}^{s} a_m w^m - \sum_{m=0}^{s} b_m w^m \ln(w)$$

$$= \sum_{m=0}^{s} a_n e^{mz} - z \sum_{m=0}^{s} b_m e^{mz}$$

Now

$$e^{mz} = 1 + mz + \frac{1}{2!}(mz)^2 + \frac{1}{3!}(mz)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(mz)^k$$

Therefor,

$$\rho(w) - \sigma(w)\ln w = \sum_{m=0}^{s} a_n \sum_{k=0}^{\infty} \frac{1}{k!}(mz)^k - z \sum_{m=0}^{s} b_m \sum_{k=0}^{\infty} \frac{1}{k!}(mz)^k$$

Compare with

$$\sum_{k=1}^{\infty} \left(\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!} m^k - \sum_{m=0}^{s} b_m \frac{y^{(k)}(t_0)}{(k-1)!} m^{k-1} \right) h^k$$

Continuing...

$$\rho(w) - \sigma'(w)\ln w = \sum_{m=0}^{s} a_n \sum_{k=0}^{\infty} \frac{1}{k!}(mz)^k - z\sum_{m=0}^{s} b_m \sum_{k=0}^{\infty} \frac{1}{k!}(mz)^k$$

$$= \sum_{m=0}^{s} a_n + \sum_{k=1}^{\infty}\sum_{m=0}^{s} a_n \frac{1}{k!}(mz)^k - \sum_{k=0}^{\infty}\sum_{m=0}^{s} b_m \frac{1}{k!} m^k z^{k+1}$$

$$= \sum_{m=0}^{s} a_n + \sum_{k=1}^{\infty}\sum_{m=0}^{s} a_n \frac{1}{k!}(mz)^k - \sum_{k=1}^{\infty}\sum_{m=0}^{s} b_m \frac{1}{(k-1)!} m^{k-1} z^k$$

$$= \underbrace{\sum_{m=0}^{s} a_n}_{\text{cond (1)}} + \underbrace{\boxed{\sum_{k=1}^{\infty}\left(\sum_{m=0}^{s} a_n \frac{1}{k!}(m\ )^k - \sum_{m=0}^{s} b_m \frac{1}{(k-1)!} m^{k-1}\right) z^k}}_{\text{cond (2)}}$$

Compare with

$$\sum_{k=1}^{\infty}\left(\sum_{m=0}^{s} a_m \frac{y^{(k)}(t_0)}{k!} m^k - \sum_{m=0}^{s} b_m \frac{y^{(k)}(t_0)}{(k-1)!} m^{k-1}\right) h^k$$

So powers of z^k vanishing is the same as h^k before...

Therefore...

$$\rho(w) - \sigma'(w)\ln w = O(z^{p+1}) \iff \tau = O(h^{p+1})$$

Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(w) - \sigma(w)\ln w = c(w-1)^{p+1} + \mathcal{O}(|w-1|^{p+2}), \qquad w \to 1. \tag{2.10}$$

$w = e^z$ $\qquad z = \ln w = \ln(1+w-1)$

$\alpha = w-1$ $\qquad O(z) = O(\ln(1+w-1)) = O(\log(1+\alpha))$

$$= O\left(\alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \cdots\right) = O(\alpha) = O(w-1)$$

What is $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$

$$1+t+t^2+t^3+\dots = \frac{1}{1-t}$$

$$1-t+t^2-t^3+\dots = \frac{1}{1+t}$$

$$\int_0^x (1-t+t^2-t^3+\dots)\,dt = \int_0^x \frac{1}{1+t}\,dt$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, = \log(1+x)$$

$$\rho(w) - \sigma(w)\ln w = O(z^{p+1}) = O(z)^{p+1} = O(w-1)^{p+1} = O((w-1)^{p+1})$$

Thus we want

$$\rho(w) - \sigma(w)\ln w = O((w-1)^{p+1})$$

Therefore...

Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(w) - \sigma(w)\ln w = c(w-1)^{p+1} + \mathcal{O}\left(|w-1|^{p+2}\right), \qquad w \to 1. \qquad (2.10)$$