$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

$$\rho(w) = \sum_{m=0}^{s} a_m w^m \quad \text{and} \quad \sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(w) - \sigma(w)\ln w = c(w-1)^{p+1} + \mathcal{O}(|w-1|^{p+2}), \qquad w \to 1. \tag{2.10}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, = \log(1+x)$$

$w = e^z$ $\qquad z = \ln w = \ln(1+w-1)$

$\alpha = w-1$ $\qquad O(z) = O(\ln(1+w-1)) = O(\log(1+\alpha))$

$$= O\left(\alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \dots\right) = O(\alpha) = O(w-1)$$

Idea: Use theorem 2.1 to maximize to order of an s step method... Bad idea...

microoptimization or premature optimization

doesn't consider the big picture and goes badly...

Main thing missing from our considerations is convergence and stability...

Consider:

$$y_{n+2} - 3y_{n+1} + 2y_n = h\left(\frac{13}{12} f(t_{n+2}, y_{n+2}) - \frac{5}{3} f(t_{n+1}, y_{n+1}) - \frac{5}{12} f(t_n, y_n)\right)$$

$$\rho(w) = w^2 - 3w + 2$$

$$\sigma(w) = \frac{13}{2} w^2 - \frac{5}{3} w - \frac{5}{12}$$

Claim is:

$$\rho(w) - \sigma(w)\ln w = O((w-1)^3)$$

So the order of the method is 2. However it is not convergent...

Consider solving the ODE

$y' = 0$ such that $y(0) = 1$

Thus $f(t,y) = 0$

solution $y(t) = 1$

$$y_{n+2} - 3y_{n+1} + 2y_n = h\left(\frac{13}{12}\cancel{f(t_{n+2},y_{n+2})} - \frac{5}{3}\cancel{f(t_{n+1},y_{n+1})} - \frac{5}{12}\cancel{f(t_n,y_n)}\right)$$

Thus $\boxed{y_{n+2} - 3y_{n+1} + 2y_n = 0}$ ← difference equation in y...

Solve it analytically

substitute $y_n = a^n$

$$a^{n+2} - 3a^{n+1} + 2a^n = 0$$

Same polynomial as ρ... →

$$a^2 - 3a + 2 = 0$$

$$(a-2)(a-1) = 0$$

$$a = 2 \quad \text{and} \quad a = 1$$

This is the same root that's bigger than 1 in the root condition..

General solution $y_n = \boxed{C_1 1^n} + C_2 2^n$

correct solution

Note because $2^n \to \infty$ as $n \to \infty$ this solution is exp. increasing whenever $c_2 \neq 0$.

Expect $c_1 = 1$ and $c_2 = 0$ gives the solution $y_n = 1$

problem due to rounding error $c_2 \approx 0$. So after a few steps there is this numerical artifact that is increasing exponentially...

This exponential artifact only gets worse as $h \to 0$. since take more steps and the initial rounding error in c_2 is always about the same., but n is larger since the steps are smaller and the approximation just gets worse and worse...

Dahlquist: root condition and equivalence theorem.

root condition: a polynomial ρ satisfies the root condition if all root w such that $\rho(w) = 0$ obey eith

① $|w| < 1$

or ② $|w| = 1$ and w is a simple root

mult of w is 1,

i.e. $\rho'(w) \neq 0$.

equivalence: The multistep method is convergent if and only if ρ satisfies the root condition.

Recall The polynomial $\rho(w) = w^2 - 3w + 2$ has the roots $w = 1$ and $w = 2$ so it doesn't satisfy the root condition and the method isn't convergent..

Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\rho(w) - \sigma(w)\ln w = c(w-1)^{p+1} + \mathcal{O}(|w-1|^{p+2}), \qquad w \to 1. \qquad (2.10)$$

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

how many parameters $2(s+1)$

but divide out by a_s and there are really only $2s+1$ parameters..

If I want $\tau_n = O(h^{p+1})$ then the method to be exact when y is a polynomial of degree p. How many parameters in a polynomial of degree p? $p+1$. actually since one can just as well consider monic polynomials, then really p parameters...

One expects that if we optimizing only for the order that one could obtain an $O(h^p) = O(h^{2s+1})$ method. Still a bad idea...

Dahlquist first barrier: If the result is actually going to be convergent then an s-step method is at most

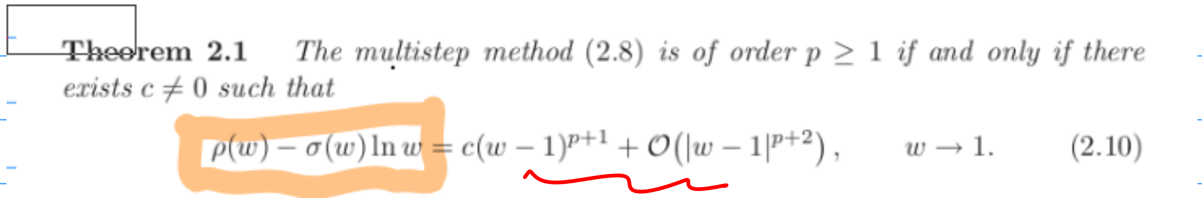
$O(h^s)$ if an explicit method

$\underbrace{O(h^{2\lfloor (s+2)/2 \rfloor})}_{\text{in practice } O(h^{s+1})}$ if implicit where $\lfloor x \rfloor$ is the greatest integer less than or equal x.

Since optimizing to order goes wrong... and we need ρ to satisfy the root condition anyway.:

New Idea... Start with ρ that satisfies the root condition and use the theorem to solve for σ.

Subject to having a convergent scheme given by a fixed ρ we optimize to order...



Theorem 2.1 *The multistep method (2.8) is of order $p \geq 1$ if and only if there exists $c \neq 0$ such that*

$$\boxed{\rho(w) - \sigma(w)\ln w} = c\underbrace{(w-1)^{p+1}}_{} + \mathcal{O}\left(|w-1|^{p+2}\right), \qquad w \to 1. \qquad (2.10)$$

Thus $$\rho(w) - \sigma(w)\ln w = O(|w-1|^{p+1})$$

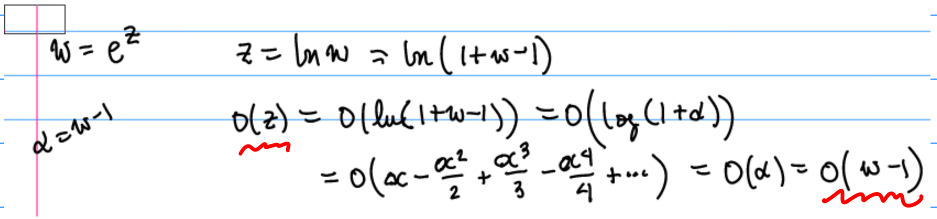
$$\sigma(w)\ln w = \rho(w) + O(|w-1|^{p+1})$$

$$\sigma(w) = \frac{\rho(w)}{\ln w} + \boxed{\frac{O(|w-1|^{p+1})}{\ln w}}$$

$$= \frac{\rho(w)}{\ln w} + O(|w-1|^{p})$$

simplify to get

recall



$w = e^z$ $\quad z = \ln w = \ln(1+w-1)$

$\alpha = w-1$ $\quad \underbrace{O(z)}_{} = O(\ln(1+w-1)) = O(\log(1+\alpha))$

$$= O\left(\alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \cdots\right) = O(\alpha) = \underbrace{O(w-1)}_{}$$

$$O(\ln w) = O(z) = O(w-1)$$

Again: Choose $\rho(w)$ then use

$$\sigma(w) = \frac{\rho(w)}{\ln(w)} + O(|w-1|^2)$$

So solve for σ.

Popular choices for ρ:

Adam's Bashforth methods...

$$\int_{t_n}^{t_{n+1}} y' = \int_{t_n}^{t_{n+1}} f(t,y)$$

$$\underbrace{y(t_{n+1}) - y(t_n)}_{\rho} = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

$$\approx \underbrace{\int_{t_n}^{t_{n+1}} \text{polynomial approximating } f.}_{\sigma}$$

Shift to obtain...

$$y(t_{n+s}) - y(t_{n+s-1}) \approx \int_{t_{n+s-1}}^{t_{n+s}} \text{polynomial for } f.$$

$$\rho(w) = w^{n+s} - w^{n+s-1} = w^{n+s-1}(w-1)$$

roots of $\rho(w)$ are $w=0$ and $w=1$

↑ root of multiplicity $n+s-1$

↑ simple root

from before AB2

$$\boxed{y_{n+1} = y_n + h\left(\frac{3}{2} f(t_n, y_n) - \frac{1}{2} f(t_{n-1}, y_{n-1})\right)}$$

shifted: $y_{n+2} - y_{n+1} \sim h \frac{3}{2}\left(f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n)\right)$

is a 2-step method that is also 2nd order...

Could also set $\rho(w) = w(w-1)$ and solve for

$$\sigma(w) = \frac{3}{2} w - \frac{1}{2}$$

If I allow σ to be quadratic one gets an implicit method of order $O(h^3)$ (Adams-Moulton method)

	$\rho(w)$	$\sigma(w)$
Adams Bashforth	$w^{s-1}(w-1)$	$\frac{\rho(w)}{\log w} + O((w-1)^q)$ explicit
Adam Moulton	〃	$\frac{\rho(w)}{\log w} + O((w-1)^{q+1})$ implicit
Nystrom	$w^{s-2}(w^2-1)$	$\frac{\rho(w)}{\log w} + O((w-1)^q)$ explicit
Milne	〃	$\frac{\rho(w)}{\log w} + O((w-1)^{q+1})$ implicit...