Backwards differentiation formula solves for $\rho(w)$ given $\sigma(w)$:

$$\rho(w) = \underbrace{\log w\, \sigma(w)}_{\text{product}} + O(|w-1|^{p+1}) \quad \text{as } w \to 1.$$

where $\sigma(w) = \beta w^s$ Note this is an implicit

Goal: Just expand everything in terms of $w-1$. Algebra to make this easier..

$$v = \frac{1}{w} \quad \text{then} \quad v \to 1 \quad \text{as} \quad w \to 1$$

Thus, $\rho\left(\frac{1}{v}\right) = \log\frac{1}{v}\, \sigma\left(\frac{1}{v}\right)$ and $\sigma\left(\frac{1}{v}\right) = \beta \frac{1}{v^s}$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, = \log(1+x)$$

$$\log\frac{1}{v} = -\log v = -\log(1+(v-1)) = -(v-1) + \frac{(v-1)^2}{2} - \frac{(v-1)^3}{3} + \frac{(v-1)^4}{4} + \dots$$

How many terms to keep. Hint: an s-step BDF has order s.

backwards differentiation formula

So take $p = s$.

$$\log\frac{1}{v} = \sum_{m=1}^{s} \frac{(-1)^m}{m}(v-1)^m + O((v-1)^{s+1})$$

$$\rho(w) = \rho\left(\frac{1}{v}\right) = \log\frac{1}{v}\,\sigma\left(\frac{1}{v}\right) = \sum_{m=1}^{s} \frac{(-1)^m}{m}\left((v-1)\frac{v}{v}\right)^m \beta\frac{1}{v^s}$$

$$= \beta \sum_{m=1}^{s} \frac{(-1)^m}{m}\left(1-\frac{1}{v}\right)^m v^m \frac{1}{v^s} + O((v-1)^{s+1})\,\beta\frac{1}{v^s}$$

$$= \beta \sum_{m=1}^{s} \frac{1}{m}(w-1)^m v^{m-s} = \underbrace{\beta \sum_{m=1}^{s} \frac{1}{m}(w-1)^m w^{s-m}}_{\text{exactly a polynomial of degree } s\text{, which was the point..}}$$

Therefore

$$\rho(w) = \dot{\beta} \sum_{m=1}^{s} \frac{1}{m} (w-1)^m w^{s-m}$$

Note that this is for the multistep method

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

$$\rho(w) = \sum_{m=0}^{s} a_m w^m \quad \text{and} \quad \sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Let's assume we've already divided through by a_s so that we can assume ρ is a monic polynomial.

Do this for definiteness so we can identify β.

What's the condition on β so that

$$\rho(w) = \dot{\beta} \sum_{m=1}^{s} \frac{1}{m} (w-1)^m w^{s-m}$$

is a monic polynomial.

leading terms in each of the summands

$$\frac{1}{m} (w-1)^m w^{s-m} = \frac{1}{m} \left(w^m + \cdots \right) w^{s-m} = \frac{1}{m} w^s + \cdots,$$

1

Therefore $$\beta \sum_{m=1}^{s} \frac{1}{m} = 1 \quad \text{or} \quad \beta = \left(\sum_{m=1}^{s} \frac{1}{m} \right)^{-1}$$

Theorem 2.4 *The polynomial (2.14) obeys the root condition and the underlying BDF method is convergent if and only if $1 \le s \le 6$.* ■

$$\rho(w) = \boxed{\beta} \sum_{m=1}^{s} \frac{1}{m} (w-1)^m w^{s-m} \qquad (2.14)$$

I.e. If $s=7$ the method is not convergent.

Recall the root condition say that
all roots w such that $\rho(w)=0$ satisfy either

① $|w|<1$

or ② $|w|=1$ and the root is simple....

Note it's not surprising that some ρ may not satisfy the root condition. That's why we usually start with a good ρ and solve for σ instead...

Chapter 3: Recall solving ODE's ...

$$y' = f(t,y)$$

First thing integrate both sides.

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t,y(t))\,dt$$

Fundamental theorems of calculus

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

Last, approximate integral on the right...

The main idea is approximating the integral... Chapter 3.1 focus on Quadrature which is the approximation of integrals... interesting on its own...

General problem given a weight $w(\tau) \geq 0$ for all τ and a function $f(\tau)$ approximate

$$\int_a^b f(\tau)\, w(\tau)\, d\tau$$

Typical weight functions

$w(\tau) = 1$

$w(\tau) = e^{at}$

$w(t) = \sqrt{1-t^2}$

but how ...

$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{i=1}^{\nu} b_i f(c_i)$$

Quadrature formula.

where the values of b_i and c_i do not depend on f

Remark the values of b_i and c_i surely depend on a, b and the weight w.

Note for trapezoid method $b_1 = b_2 = \frac{b-a}{2}$

and $c_1 = a$ and $c_2 = b$.

Peano kernel theorem: $\rightarrow$ Appendix A.

linear in b_i's on the right...

$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{i=1}^{\nu} b_i f(c_i)$$

If the c_i's have already been determined, finding good b_i's involves linear equations..

For each different f you get a different equation... there are lots of linear equations and ν parameters b_i to optimize.... use least squares to find the b_i's...

Take a subset of possible f's and optimize over those...

Simplicity use exactly ν different functions so the linear system is not over constrained...

Even simpler use the ν polynomials

$$\{1, t, t^2, \ldots, t^{\nu-1}\}$$

This is also a good choice because polynomials are useful for approximating other functions...

Thus given the values of c_i (which we can choose arbitrarily at this point provided they are all different) one can solve for b_i's using

$$\int_a^b \tau^m w(\tau)\,d\tau \approx \sum_{i=1}^{\nu} b_i c_i^m = Vb$$

for $m = 0, 1, \ldots, \nu-1$.

Solve for b_i's using Vandermond matrix

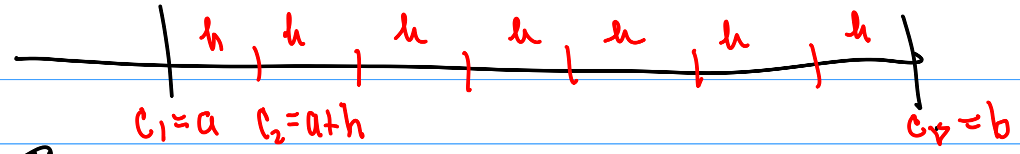
$$V = \begin{array}{c} \begin{array}{cccccc} m=0 & m=1 & & & & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ c_1 & c_2 & c_3 & & c_\nu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1^{\nu-1} & c_2^{\nu-1} & c_3^{\nu-1} & \cdots & c_\nu^{\nu-1} \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\nu \end{bmatrix} \qquad y = \begin{bmatrix} \int_a^b w(\tau)\,d\tau \\ \int_a^b \tau\, w(\tau)\,d\tau \\ \vdots \\ \int_a^b \tau^{\nu-1} w(\tau)\,d\tau \end{bmatrix}$$

and solving $Vb = y$ for b since V is square in this case and invertible.. (the b_i's are uniquely determined.)

This gives a good choice for b_i's given the c_i's, but what is a good way to choose the c_i's?

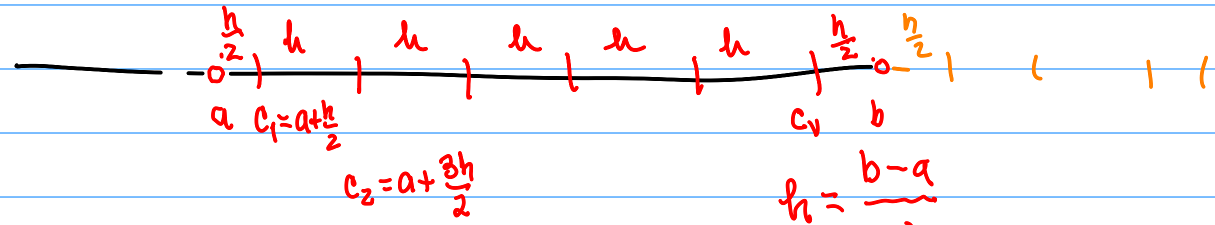
$$\int_a^b f(\tau)\, w(\tau)\,d\tau \approx \sum_{i=1}^{\nu} b_i f(c_i)$$

$\nu = 8$



$h = \frac{b-a}{\nu-1}$

equally spaced including the endpoints... $\boxed{\text{Closed Newton-Cotes}}$ formula.



$h = \frac{b-a}{\nu}$

Open formula

$\boxed{\text{Open Newton-Cotes}}$.

How to best choose the c_i's?

$\boxed{\text{Gauss quadrature methods.}}$