$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{i=1}^{\nu} b_i f(c_i)$$

linear in $b_i$'s on the right...

If the $c_i$'s have already been determined, finding good $b_i$'s involves linear equations...

Use polynomials to solve for the $b_i$'s...

$$\int_a^b \tau^m w(\tau)\, d\tau = \sum_{i=1}^{\nu} b_i c_i^m \qquad \text{for } m = 0, 1, \ldots, \nu-1$$

Stop at $\nu-1$ to have same # of equations as unknowns.

If there are exactly same # of unknowns and vbls can solve for equality.

Theoretical way to find the $b_i$'s... using Lagrange basis functions.

$$p_j(t) = \prod_{\substack{k=1 \\ k\neq j}}^{\nu} \frac{t - c_k}{c_j - c_k}$$

Note:

$$p_j(c_\ell) = \begin{cases} 1 & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell \end{cases}$$

and $p_j$ is a polynomial of degree $\nu-1$

If $g(t)$ is an arbitrary polynomial of degree $\nu-1$ then

$$g(t) = \sum_{j=1}^{\nu} p_j(t)\, g(c_j)$$

since $\nu$ points uniquely determine a polynomial of degree $\nu-1$.

Set $g(t) = t^m$ for some fixed $m$ where $0 \le m \le \nu-1$.

$$t^m = \sum_{j=1}^{\nu} p_j(t)\, c_j^m$$

complicated way to write something really simple...

$$\int_a^b \tau^m w(\tau)\,d\tau \approx \int_a^b \sum_{j=1}^{\nu} P_j(\tau)\, c_j^m\, w(\tau)\,d\tau$$

$$\approx \sum_{j=1}^{\nu}\left(\int_a^b P_j(\tau) w(\tau)\,d\tau\right) c_j^m \approx \sum_{i=1}^{\nu} b_i\, c_i^m$$

have to be the same.

Therefore...

$$b_j = \int_a^b P_j(\tau) w(\tau)\,d\tau$$

used orthogonality and uniqueness to get this result.

This is the end of the story if the $c_i$'s are given --- and sometimes we just want equally spaced points in the first place so how to choose the $c_i$'s is already set..

what is the order?

However... If we want to optimize to order of convergence of this formula the we need to choose the $c_i$'s differently (carefully).

Gauss Quadrature methods (maybe not until next time)

Peano Kernel Theorem... Interpolating polynomial theorem

Taylors theorem:

$$f(t) = T_n(t) + R_n(t)$$

Taylor polynomial of degree $n$

where $R_n(t) = \frac{1}{(n+1)!}(t-t_0)^{n+1} f^{(n+1)}(\xi)$ for some $\xi$ between $t$ and $t_0$

Alternatively

taylor polynomial of degree $\nu-1$

$$\boxed{f(t) = T_{\nu-1}(t) + R_{\nu-1}(t)}$$

$$R_{\nu-1}(t) = \frac{1}{\nu!}(t-t_0)^{\nu} f^{(\nu)}(\xi) \quad \text{for some } \xi \text{ between } t \text{ and } t_0$$

Given a function $f$ we approximate using the interpolating polynomial $g$.

$$f(t) \approx g(t) = \sum_{i=1}^{\nu} p_i(t) f(c_i)$$

Lagrange basis functions.

Define the error

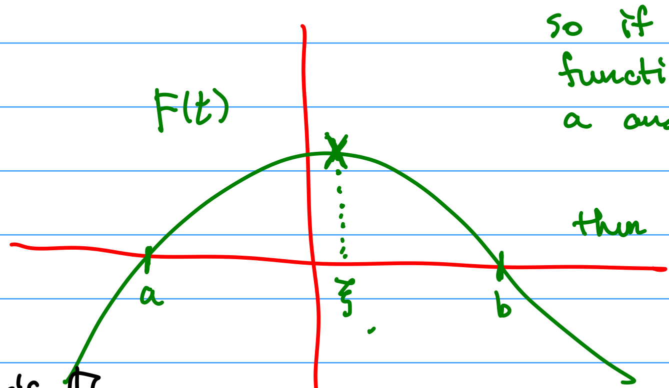
$$\boxed{f(t) = g(t) + E(t)}$$

interpolating polynomial of degree $\nu-1$

Thus, $f(c_i) = g(c_i)$ for $i = 1, \dots, \nu$.

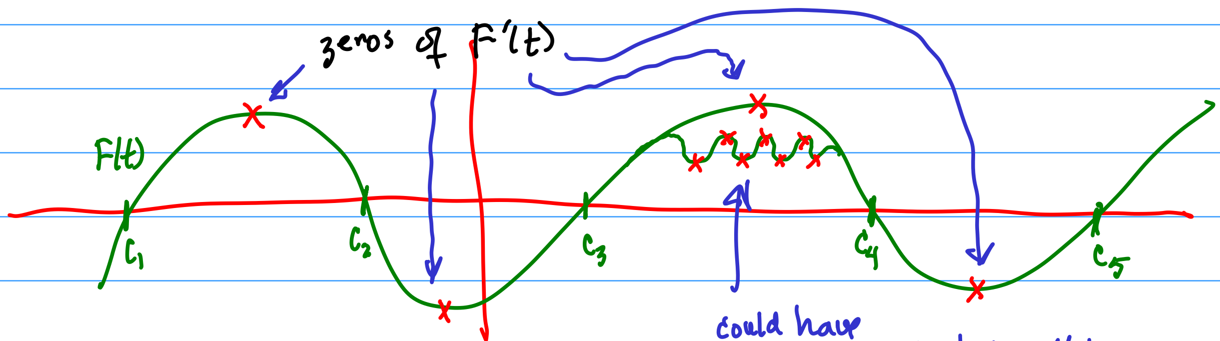
$E(t) = f(t) - g(t)$ but how to bound this?

## Recall Rolle's theorem




So if a differentiable function has roots at $a$ and $b$ so $F(a)=0$ $F(b)=0$ then somewhere between $a$ and $b$ we have $F'(\xi)=0$

General Rolle's Theorem




could have extra zeros, but in this case at least 4 zeros are guaranteed...

Just like with the simple rolles theorem the zeros of the derivative are between the zeros of the original function.

Conclusion if $F(t)$ has $n$ zeros

then $F'(t)$ has at least $n-1$ zeros

Define $E(t) = f(t) - g(t)$ note that $E(t)$ has $\nu$ zeros...

$E'(t)$ has $\nu-1$ zeros...

$E^{(2)}(t)$ has $\nu-2$ zero...

To estimate $E$ I want to care it with another function that has even more zeros...

$$F(t) = E(t) - \alpha q(t) \quad \text{where } q(t) = (t-c_1)(t-c_2)\cdots(t-c_\nu)$$

Note $F(c_i) = E(c_i) - \alpha q(c_i) = 0$ for $i = 1, \ldots, \nu$.

To make another root adjust $\alpha$. Suppose I want to make another root at $t_*$. Then $t_* \neq c_i$ for any $i$ and $q(t_*) \neq 0$.

$$0 = F(t_*) = E(t_*) - \alpha q(t_*) = 0$$

Solve for $\alpha$: $\alpha = \dfrac{E(t_*)}{q(t_*)}$

Now $F(t)$ has the roots $\boxed{c_1, c_2, \ldots, c_\nu \text{ and also } t_*}$

$F(t)$ has $\nu+1$ roots.

$F'(t)$ has $\nu$ roots

$F''(t)$ has $\nu-1$ roots

$F^{(\nu)}(t)$ has 1 root

and the root $\xi$ is between the roots of the origional function $F$, in particular
$\min\{c_1, \ldots, c_\nu, t_*\} \le \xi \le \max\{c_1, \ldots, c_\nu, t_*\}$

Solve for that root ... $F^{(\nu)}(\xi) = 0$

call the root $\xi$.

$$F^{(\nu)}(\xi) = E^{(\nu)}(\xi) - \frac{E(t_*)}{q(t_*)} q^{(\nu)}(\xi) = 0$$

$$q^{(\nu)}(t) = \frac{d^\nu}{dt^\nu}\left[(t-c_1)(t-c_2)\cdots(t-c_\nu)\right] = \nu!$$

$$E^{(\nu)}(t) = f^{(\nu)}(t) - g^{(\nu)}(t) = f^{(\nu)}(t)$$

interpolating polynomial of degree $\nu-1$

Therefore,

$$f^{(\nu)}(\xi) - \frac{E(t_*)}{q(t_*)}\nu! = 0$$

so

$$E(t_*) = \frac{q(t_*)\, f^{(\nu)}(\xi)}{\nu!}$$

Compare to remainder in Taylor theorem

$$R_{\nu-1}(t) = \frac{1}{\nu!}(t-t_0)^\nu f^{(\nu)}(\xi)$$

In summary:

$$f(t_*) = g(t_*) + \frac{q(t_*) f^{(\nu)}(\xi)}{\nu!}$$

for some $\xi$ between the roots of $F$.

$$\min\{c_1,\dots,c_\nu,t_*\} \le \xi \le \max\{c_1,\dots,c_\nu,t_*\}$$

Theorem. Suppose $f$ has $\nu$ derivatives, then.

$$f(t) = g(t) + \frac{\prod_{i=1}^{\nu}(t-c_i)}{\nu!}\, f^{(\nu)}(\xi)$$

where $g(t)$ is the interpolating polynomial through $(c_i, f(c_i))$ for $i=1,\dots\nu$. and $\xi$ is some value such that $\min\{c_1,\dots,c_\nu,t\} \le \xi \le \max\{c_1,\dots,c_\nu,t\}$.

Use this theorem to estimate the errors in

$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{i=1}^{\nu} b_i f(c_i)$$

linear

Note first for any polynomial $g(x)$ of degree $\nu-1$ that

$$\int_a^b g(\tau)\,w(\tau)\,d\tau = \sum_{i=1}^{\nu} b_i\, g(c_i).$$

$$\text{Error} = \left| \sum_{i=1}^{\nu} b_i f(c_i) - \int_a^b f(\tau) w(\tau)\, d\tau \right|$$

Let $g$ be the interpolating polynomial of $f$

$$\Bigg| \underbrace{\sum_{i=1}^{\nu} b_i f(c_i) - \sum_{i=1}^{\nu} b_i\, g(c_i)}_{\substack{\text{are the same} \\ \text{since } f(c_i) = g(c_i)}} + \int_a^b g(\tau) w(\tau)\, d\tau - \int_a^b f(\tau) w(\tau)\, d\tau \Bigg|$$

$$\leq \int_a^b \left| g(\tau) - f(\tau) \right| \left| w(\tau) \right| d\tau$$

$$= \int_a^b \left| \frac{\prod_{i=1}^{\nu} (\tau - c_i)}{\nu!} f^{(\nu)}(\xi(\tau)) \right| \left| w(\tau) \right| d\tau$$

$$\leq \max_{a \leq \xi \leq b} \left| f^{(\nu)}(\xi) \right| \cdot \underbrace{\int_a^b \left| \frac{q(\tau)}{\nu!} w(\tau) \right| d\tau}_{\text{Some constant (doesn't depend on } f\text{)}}$$

assume $\tau, c_1, c_2, \ldots c_\nu$ are all inside the interval $[a,b]$.

$$\leq c \max_{a \leq \xi \leq b} \left| f^{(\nu)}(\xi) \right|$$

We have shown this error formula for the quadrature method

to prove, e.g. by using the Peano kernel theorem (see A.2.2.6), that, for every function $f$ with $p$ smooth derivatives,

$$\left| \int_a^b f(\tau)\omega(\tau)\,d\tau - \sum_{j=1}^{\nu} b_j f(c_j) \right| \leq c \max_{a \leq t \leq b} \left| f^{(p)}(t) \right|,$$