Gram-Schmidt:

$z_0 = u_0$ $\qquad$ $p_0 = z_0 / \|z_0\|$

$z_1 = u_1 - \langle p_0, u_1 \rangle p_0$ $\qquad$ $p_1 = z_1 / \|z_1\|$

$z_2 = u_2 - \langle p_0, u_2 \rangle p_0 + \langle p_1, u_2 \rangle p_1$ $\qquad$ $p_2 = z_2 / \|z_2\|$

$\vdots$

$z_{\nu-1} = u_{\nu-1} - \langle p_0, u_{\nu-1} \rangle p_0 - \cdots - \langle p_{\nu-2}, u_{\nu-1} \rangle p_{\nu-2}$

$p_{\nu-1} = z_{\nu-1} / \|z_{\nu-1}\|$

Trying to find a Quadrature formula like this:

$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{j=1}^{\nu} b_j\, f(c_j)$$

weight

So we assume a, b and w are given. Easiest example let $a = 0$, $b = 1$ and $w(\tau) = 1$.

$$\int_0^1 f(\tau)\, d\tau \approx \sum_{j=1}^{\nu} b_j f(c_j)$$

In general the inner product is

$$\langle f, g \rangle = \int_a^b f(\tau) g(\tau)\, w(\tau)\, d\tau$$

Inner product

$$\langle p, q \rangle = \int_0^1 p(\tau) q(\tau)\, d\tau$$

$$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{\int_0^1 (p(\tau))^2\, d\tau}$$

Gram-Schmidt: Polynomial basis for polynomials of degree up to ν.

$u_0(\tau) = 1$, $u_1(\tau) = \tau$, $u_2(\tau) = \tau^2$, $\ldots$, $u_\nu(\tau) = \tau^\nu$

First step.

$Z_0 = u_0$ $\qquad$ $p_0 = Z_0 / \|Z_0\|$

$z_0 = 1$ $\qquad$ $p_0 = 1/\|1\| = 1/1 = 1$

$$\|1\| = \sqrt{\int_0^1 1^2 \, d\tau} = 1$$

Second step

$z_1 = u_1 - (p_0, u_1) p_0$

$z_1 = \tau - (1, \tau) 1 = \tau - \frac{1}{2}$ $\qquad$ $p_1 = \frac{z_1}{\|z_1\|} = \frac{\tau - \frac{1}{2}}{\left(\frac{1}{2\sqrt{3}}\right)} = 2\sqrt{3}\left(\tau - \frac{1}{2}\right)$

$$(1, \tau) = \int_0^1 1 \cdot \tau \, d\tau = \frac{\tau^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\|z_1\| = \left\|\tau - \frac{1}{2}\right\| = \sqrt{\int_0^1 \left(\tau - \frac{1}{2}\right)^2 d\tau} = \sqrt{\frac{1}{3}\left(\tau - \frac{1}{2}\right)^3 \Big|_0^1}$$

$$= \sqrt{\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{3}\left(-\frac{1}{2}\right)^3} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$$

Third step:

$z_2 = u_2 - \langle p_0, u_2 \rangle p_0 - \langle p_1, u_2 \rangle p_1$

$$z_2 = \tau^2 - \langle 1, \tau^2 \rangle 1 - \left\langle 2\sqrt{3}\left(\tau - \frac{1}{2}\right), \tau^2 \right\rangle 2\sqrt{3}\left(\tau - \frac{1}{2}\right)$$

$$\langle 1, \tau^2 \rangle = \int_0^1 \tau^2 \, d\tau = \frac{1}{3}\tau^3 \Big|_0^1 = \frac{1}{3}$$

$$\left\langle \tau - \frac{1}{2}, \tau^2 \right\rangle = \int_0^1 \left(\tau - \frac{1}{2}\right) \tau^2 \, d\tau = \int_0^1 \left(\tau^3 - \frac{1}{2}\tau^2\right) d\tau$$

$$= \frac{1}{4}\tau^4 - \frac{1}{6}\tau^3 \Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$$z_2 = \tau^2 - \frac{1}{3} - 12 \cdot \frac{1}{12}\left(\tau - \frac{1}{2}\right) = \tau^2 - \tau - \frac{1}{3} + \frac{1}{2} = \tau^2 - \tau + \frac{1}{6}$$

$$P_2 = \frac{z_2}{\|z_2\|} = \frac{\underbrace{\tau^2 - \tau + \frac{1}{6}}}{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}}$$

$$\|z_2\|^2 = \|\tau^2 - \tau + \tfrac{1}{6}\|^2 = \int_0^1 (\tau^2 - \tau + \tfrac{1}{6})^2 \, d\tau$$

$$= \int_0^1 (\tau^4 - 2\tau^3 + \tfrac{4}{3}\tau^2 - \tfrac{1}{3}\tau + \tfrac{1}{36}) \, d\tau$$

$$= \tfrac{1}{5}\tau^5 - \tfrac{2}{4}\tau^4 + \tfrac{4}{9}\tau^3 - \tfrac{1}{6}\tau^2 + \tfrac{1}{36}\tau \Big|_0^1$$

$$= \tfrac{1}{5} - \tfrac{1}{2} + \tfrac{4}{9} - \tfrac{1}{6} + \tfrac{1}{36} =$$

Back to Gaussian Quadrature...

① Now choose c_i's to be the roots of $p_\nu(\tau)$

↑ ort

the root of P_1

Try $P_1(\tau) = 2\sqrt{3}(\tau - \frac{1}{2}) = 0$ $\qquad c_1 = \frac{1}{2}$

② Choose the b_i's so that

$$\int_a^b \tau^m w(\tau) d\tau = \sum_{j=1}^{\nu} b_j c_j^m \qquad \text{for } m = 0, 1, \ldots, \nu - 1$$

$$\int_0^1 \tau^m d\tau = b_1 c_1^m \qquad \text{for } m = 0$$

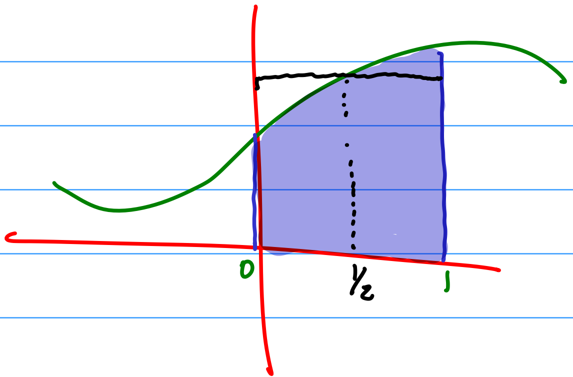
$$\int_0^1 d\tau = b_1 \qquad b_1 = 1.$$

Thus

$$\int_0^1 f(\tau) d\tau \approx b_1 f(c_1)$$

or

$$\int_0^1 f(\tau) d\tau \approx f(\tfrac{1}{2})$$



← midpoint rule for approximating an integral...

The next Gaussian Quadrature formulas.

① Now choose c_i's to be the roots of $p_\nu(\tau)$

ort

Try $P_2(\tau) = \dfrac{\tau^2 - \tau + \frac{1}{6}}{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = 0$

$$\tau^2 - \tau + \frac{1}{6} = 0$$

$$\left(\tau - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{6} = 0$$

$$\left(\tau - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$$\tau - \frac{1}{2} = \pm \frac{1}{\sqrt{12}}$$

$$\tau = \frac{1}{2} \pm \frac{1}{\sqrt{12}} \qquad c_1 = \frac{1}{2} - \frac{1}{\sqrt{12}}, \quad c_2 = \frac{1}{2} + \frac{1}{\sqrt{12}}$$

② Choose the b_i's so that

$$\int_a^b \tau^m w(\tau)\,d\tau = \boxed{\sum_{j=1}^{\nu} b_j c_j^m} \quad \text{for } m = 0, 1, \dots, \nu - 1$$

$$\begin{cases} \boxed{1 = \int_0^1 d\tau = b_1 \cdot \overset{c_1^0}{1} + b_2 \cdot \overset{c_2^0}{1}} & (m=0) \\ \frac{1}{2} = \int_0^1 \tau\,d\tau = b_1 c_1 + b_2 c_2 & (m=1) \end{cases}$$

$$\begin{cases} \boxed{b_1 + b_2 = 1} \\ b_1\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + b_2\left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right) = \frac{1}{2} \end{cases}$$

$$\begin{cases} \boxed{b_1 + b_2 = 1} \\ \frac{1}{2}(b_1 + b_2) + \frac{1}{\sqrt{12}}(-b_1 + b_2) = \frac{1}{2} \end{cases}$$

$$\frac{1}{2} + \frac{1}{\sqrt{12}}(-b_1 + b_2) = \frac{1}{2}$$

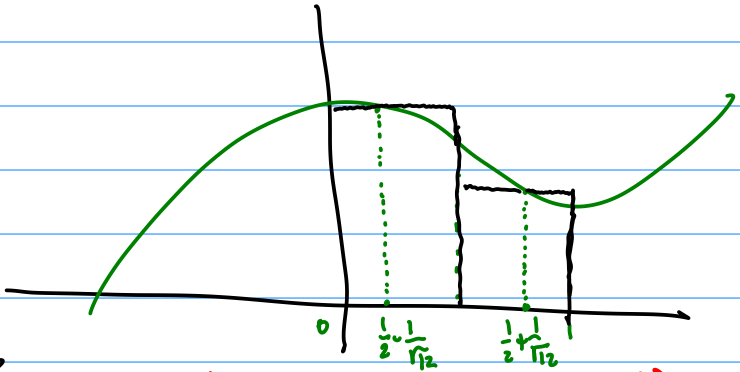
$$b_1 = b_2 \qquad 2b_1 = 1 \qquad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}$$

Thus,

$$\int_0^1 f(\tau)\,d\tau \approx \sum_{i=1}^{2} b_i f(c_i)$$

is

$$\int_0^1 f(\tau)\,d\tau \approx \frac{1}{2} f\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right)$$



$\nu = 2$ since we used the roots of P_2.

(Should be order 4 method)

Why is this a good method?, It's looks strange...

Try to prove this...

The conclusion: The resulting Quadrature method is of order 2ν.

How good is the approximation (3.2)? Suppose that the quadrature matches the integral exactly whenever f is an arbitrary polynomial of degree $p-1$. It is then easy

where the constant $c > 0$ is independent of f. Such a quadrature formula is said to be of *order* p.

Thus, I need show

$$\int_a^b p(\tau)w(\tau)\,d\tau = \sum_{i=1}^{\nu} b_i p(c_i)$$

for every polynomial p of degree less or equal $2\nu - 1$.

Let p be a polynomial of degree $2\nu-1$.

Recall c_i are the roots of the orthogonal polynomial P_ν of degree ν.

$$P_\nu \overset{q}{\overline{\big)\, p}} \quad + \text{ remainder } r.$$

Thus

$$\underset{\text{degree } 2\nu-1}{p(\tau)} = \underset{\text{degree } \nu}{P_\nu(\tau)}\, \underset{\text{degree } \nu-1}{q(\tau)} + \underset{\text{degree } \nu-1}{r(\tau)}$$

Plug in on the left

$$\int_a^b p(\tau) w(\tau)\, d\tau = \int_a^b \big(P_\nu(\tau) q(\tau) + r(\tau)\big) w(\tau)\, d\tau$$

$$= \int_a^b P_\nu(\tau) \underbrace{q(\tau) w(\tau)\, d\tau}_{\text{dot product}} + \int_a^b r(\tau) w(\tau)\, d\tau$$

$$= \langle \underset{\text{orthogonal polynomial of degree } \nu}{P_\nu}, \underset{\text{arbitrary poly of degree } \nu-1}{q} \rangle + \int_a^b r(\tau) w(\tau)\, d\tau$$

Gram-Schmidt implies the orthogonal poly of degree ν is orthogonal to all polynomials of lesser degree

Thus $\langle P_\nu, q \rangle = 0$

$$= \int_a^b r(\tau) w(\tau)\, d\tau$$

Plug in on the right.

$$\sum_{i=1}^{\nu} b_i\, p(c_i) = \sum_{i=1}^{\nu} b_i \big(P_\nu(c_i)\, q(c_i) + r(c_i)\big)$$

$$= \sum_{i=1}^{\nu} b_i P_\nu(c_i) q(c_i) + \sum_{i=1}^{\nu} b_i r(c_i)$$

Recall c_i are the roots of the orthogonal polynomial P_ν of degree ν. Thus, $P_\nu(c_i) = 0$ by definition of the c_i's.

Thus,

polynomial of degree ν-1

$$\sum_{i=1}^{\nu} b_i p(c_i) = \sum_{i=1}^{\nu} b_i r(c_i)$$

$$= \int_a^b r(\tau) w(\tau) d\tau$$

The b_i's we chosen so the quadrature formula was exact for polynomials of degree ν-1 or less

Thus
$$\sum_{i=1}^{\nu} b_i r(c_i) = \int_a^b r(\tau) w(\tau) d\tau$$
from how the b_i's are chosen..

Conclusion:

$$\sum_{i=1}^{\nu} b_i p(c_i) = \int_a^b p(\tau) w(\tau) d\tau$$

for all polynomials p of degree 2ν-1. This means the formula is order 2ν.