$y' = Ay$ where $y(t) \in \mathbb{R}^2$ and $A = \begin{bmatrix} -100 & 1 \\ 0 & -\frac{1}{10} \end{bmatrix}$

linear system of ODE's.

$$\begin{cases} y_1' = -100y_1 + y_2 \\ y_2' = -\frac{1}{10} y_2 \end{cases}$$

written explicitly this system

Euler's approximation:

$$y_n = (I + hA)^n y_0$$

Exact solution:

$$y(t) = e^{At} y_0$$

Simplify these two formula using the eigenvalue-eigenvector factorization of A.

$$A = VDV^{-1} \qquad V = \begin{bmatrix} 1 & 1 \\ 0 & \frac{999}{10} \end{bmatrix} \qquad D = \begin{bmatrix} -100 & 0 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

and any multiple

and obtained the explicit formula:

$$y_n = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 - h100)^n + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} \left(1 - \frac{h}{10}\right)^n$$

$$y(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-100t} + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} e^{-\frac{1}{10}t}$$

Look at exact solution to see qualitatively what's supposed to happen

- Convergence is what happens when t is fixed and $h \to 0$
- Stability is what happens when h is fixed and $t \to \infty$.

Exact solution as $t \to \infty$

$$y(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-100t} + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} e^{-\frac{1}{10}t} \to 0 \quad \text{as } t \to \infty$$

exponentially decrease to zero...

approximate solution as $t \to \infty$ then $n \to \infty$

$$y_n \approx c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1-h100)^n + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} \left(1-\frac{h}{10}\right)^n$$

need to know whether

$$\underbrace{|1-h100|<1}_{\text{solve this for } h} \qquad \text{and} \qquad \underbrace{\left|1-\frac{h}{10}\right|<1}_{\text{solve for } h}$$

$$-1 < h100 - 1 < 1 \qquad\qquad -1 < \frac{h}{10} - 1 < 1$$

$$0 < h100 < 2 \qquad\qquad 0 < \frac{h}{10} < 2$$

$$\boxed{0 < h < \frac{1}{50}} \qquad\qquad \boxed{0 < h < 20}$$

fast — two conditions ... — slow

Stiffness has to do with two different time scales in the problem. In order to have the correct qualitative behavior as $t \to \infty$ need to satisfy the fastest timescale.

In general then $h < \frac{1}{50}$.

recall that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c = V^{-1} y_0$$

Suppose the initial condition y_0 was such that $c_1 = 0$.

$$y_n = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 - h100)^n + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} \left(1 - \frac{h}{10}\right)^n$$

$$y(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-100t} + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} e^{-\frac{1}{10}t}$$

In this case one expects the solution to exhibit only the slow scale motion (i.e. slow decay to zero)

The approximation y_n is computed with Euler's method

$$y_{n+1} = y_n + h f(y_n) = y_n + h A y_n$$

Not with the simplified formula. Thus the fast and slow motion hasn't been explicitly decoupled using the eigenvalues and eigenvectors. Thus rounding errors will mix bits of the fast and slow directions together as the computation proceeds...

If y_0 is such that $c_1 = 0$ exactly, on the computer is only approximately zero due to rounding. Even if y_0 has no fast component in the beginning after some iterations y_n will again have $c_1 \neq 0$.

Thus both constraints

$$\boxed{\underset{\text{fast}}{0 < h < \frac{1}{50}}} \qquad \text{two conditions ...} \qquad \boxed{\underset{\text{slow}}{0 < h < 20}}$$

need to be satisfied whether or not the solution has any fast decay in it or not.

Trapezoid method:

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}\right)$$

- this is 2nd order so already better than Euler in one way.
- Implicit makes it worse
- What about stability?

I.e. how large can I take h and still have the correct qualitative behavior. $y_n \to 0$ as $n \to \infty$.

ODE:

$$y' = Ay$$

Trapezoid method in this case

$$y_{n+1} = y_n + h\left(\frac{Ay_n + Ay_{n+1}}{2}\right)$$

$$\left(I - \frac{h}{2}A\right) y_{n+1} \approx \left(I + \frac{h}{2}A\right) y_n$$

$$y_{n+1} \approx \left(I - \frac{h}{2}A\right)^{-1} \left(I + \frac{h}{2}A\right) y_n$$

recall

$$A = VDV^{-1} \qquad V = \begin{bmatrix} 1 & 1 \\ 0 & \frac{999}{10} \end{bmatrix} \qquad D = \begin{bmatrix} -100 & 0 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

and any multiple

$$y_{n+1} = \underbrace{V\left(1 - \frac{h}{2}D\right)^{-1} V^{-1}}_{\text{seems fishy}} V\left(1 + \frac{h}{2}D\right) V^{-1} y_n$$

check this... ☑

$$\left(I - \frac{h}{2}A\right)^{-1} = B$$

$$I - \frac{h}{2}A = B^{-1}$$

$$VV^{-1} - \frac{h}{2}VDV^{-1} = B^{-1}$$

$$V\left(I - \frac{h}{2}D\right)V^{-1} = B^{-1}$$

$$B = \left(V\left(I - \frac{h}{2}D\right)V^{-1}\right)^{-1} = \left(V^{-1}\right)^{-1}\left(1 - \frac{h}{2}D\right)^{-1} V^{-1}$$

$$= V\left(I - \frac{h}{2}D\right)^{-1} V^{-1} \quad ☑$$

Therefore,

$$y_{n+1} = V\left(1-\frac{h}{2}D\right)^{-1}\cancel{V^{-1}}\,\cancel{V}\left(1+\frac{h}{2}D\right)V^{-1}y_n$$

$$= V\left(1-\frac{h}{2}D\right)^{-1}\left(1+\frac{h}{2}D\right)V^{-1}y_n$$

$$D = \begin{bmatrix} -100 & 0 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

$$= V\begin{bmatrix} 1+\frac{h}{2}100 & 0 \\ 0 & 1+\frac{h}{2}\cdot\frac{1}{10} \end{bmatrix}^{-1}\begin{bmatrix} 1-\frac{h}{2}100 & 0 \\ 0 & 1-\frac{h}{2}\cdot\frac{1}{10} \end{bmatrix}V^{-1}y_n$$

$$= V\begin{bmatrix} \frac{1}{1+h50} & 0 \\ 0 & \frac{1}{1+\frac{h}{20}} \end{bmatrix}\begin{bmatrix} 1-h50 & 0 \\ 0 & 1-\frac{h}{20} \end{bmatrix}V^{-1}y_n$$

$$y_{n+1} = V\begin{bmatrix} \frac{1-h50}{1+h50} & 0 \\ 0 & \frac{20-h}{20+h} \end{bmatrix}V^{-1}y_n$$

by induction... $c = V^{-1}y_0$

$$y_n = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\left(\frac{1-h50}{1+h50}\right)^n + c_2\begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix}\left(\frac{20-h}{20+h}\right)^n$$

need $\left|\frac{1-h50}{1+h50}\right| < 1$ and $\left|\frac{20-h}{20+h}\right| < 1$

in order that $y_n \to 0$ as $t \to \infty$.

This is true for any $h > 0$. Thus the method is very stable...

of course when h is large you get a bad approximation but sometimes that what you want.

Also in y_0 is such that $c_0 = 0$ (approximately after rounding. Then no matter what h is used the fast component of the solution remains essentially zero for all $t \to \infty$.

Thus if the initial condition only has slow scales in it one doesn't have to worry about fast scales suddenly appearing out of nowhere and becoming unstable.

Note A was made up, but similar things happen in discretizations of PDEs involving dissipation, for example, like the heat equation, fluid flow, etc.

Section 4.2: Stability

Work with the scalar equation

$$y' = \lambda y, \qquad y(0) = 1$$

since eigenvalues and eigenvectors imply the linear systems are the same anyway.

When does the numerical approximation $y_n \to 0$ as $n \to \infty$. Compare to when the exact solution $y(t) \to 0$ as $t \to \infty$

Note λ might be complex.

Exact solution: $y(t) = e^{\lambda t}$

For what values of λ does $y(t) \to 0$ as $t \to \infty$?

$$\lambda = \alpha + i\beta$$

$$e^{\lambda t} = e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$$

$$= e^{\alpha t}\left(\cos\beta t + i \sin\beta t\right) \to 0 \quad \text{as } t \to \infty$$

provided $\alpha < 0$

or real $\lambda < 0$

Euler's method since $y' = \lambda y$ then $f(t,y) = \lambda y$ and

$$y_{n+1} = y_n + h f(t_n, y_n) = y_n + h\lambda y_n$$

$$y_{n+1} = (1 + h\lambda) y_n$$

by induction

$$y_n = (1 + h\lambda)^n y_0 = (1 + h\lambda)^n$$

when does $y_n \to 0$?

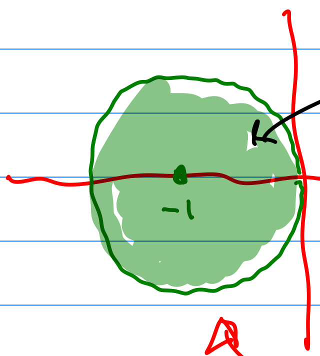
Thinking about what values of λ and what value of h together

so set $z = h\lambda$

center

When is $|1 + z| < 1$. radius

circle of radius 1 centered at -1



if λh is inside this green disk then $y_n \to 0$ as $n \to \infty$

Linear Stability domain of Euler's method