Linear stability:

Given a numeric scheme, let y_n be the approximation to the ODE

$$y' = \lambda y \quad \text{with} \quad y(0) = 1$$

Here $y_n \approx y(t_n)$ where $t_n = t_0 + h n$ and $t_0 = 0$.

The linear stability domain of that numeric scheme is

$$\mathcal{D} = \{ z \in \mathbb{C} : y_n \to 0 \quad \text{as } n \to \infty \quad \text{and } h\lambda = z \}$$

One more example: Compute $\mathcal{D}$ for the Trapezoid rule:

$$y_{n+1} = y_n + h \, \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$

Since

$$y' = \lambda y \quad \text{with} \quad y(0) = 1$$

then $f(t,y) = \lambda y$ and

$$y_{n+1} = y_n + h \, \frac{\lambda y_n + \lambda y_{n+1}}{2}$$

Simplify

$$y_{n+1}\left(1 - \frac{h\lambda}{2}\right) = y_n\left(1 + \frac{h\lambda}{2}\right)$$

$$y_{n+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \, y_n$$

Therefore, by induction

$$y_n = \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^n y_0 = \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^n$$

The linear stability domain

$$\mathcal{D} = \left\{ z \in \mathbb{C} : y_n \to 0 \quad \text{as } n \to \infty \quad \text{and } h\lambda = z \right\}$$

$$= \left\{ z \in \mathbb{C} : \left| \frac{1+z/2}{1-z/2} \right| < 1 \right\}$$

Solve the inequality

$$\left| \frac{1+z/2}{1-z/2} \right| < 1$$

Since $z \in \mathbb{C}$ then

$|z| = \sqrt{z\bar{z}}$

where

$\overline{a+ib} = a-ib$.

$|z|^2 = z\bar{z}$

$$|1+z/2|^2 < |1-z/2|^2$$

$$\left(1+\frac{z}{2}\right)\left(1+\frac{\bar{z}}{2}\right) < \left(1-\frac{z}{2}\right)\left(1-\frac{\bar{z}}{2}\right)$$

$$1+\frac{\bar{z}}{2}+\frac{z}{2}+\frac{|z|^2}{4} < 1-\frac{\bar{z}}{2}-\frac{z}{2}+\frac{|z|^2}{4}$$

$$\bar{z}+z < 0$$

$$\overline{a+ib} + a+ib < 0$$

$$a-ib + a+ib < 0$$

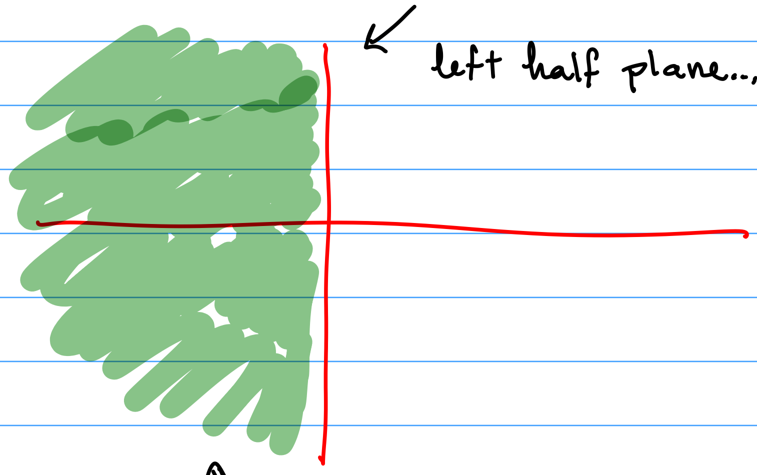
actually a real number so the inequality makes sense

$$2a < 0$$

$$\boxed{\operatorname{Re} z < 0}$$

Thus

$$\mathcal{D} = \{ z \in \mathbb{C} \; : \; \operatorname{Re} z < 0 \}$$



left half plane...,

when z is here then $y_n \to 0$ as $n \to \infty$
$z = h\lambda$ and $h > 0$ so this is equivalent
to saying $y_n \to 0$ when $\operatorname{Re} \lambda < 0$.

Exact solution of

$$y' = \lambda y \quad \text{with} \quad y(0) = 1$$

is $\quad y(t) = e^{\lambda t}$

and $\quad y(t) \to 0$ as $t \to \infty$ exactly when $\operatorname{Re} \lambda < 0$.

Since the stability domain of the Trapezoid method includes the left half plane then

$$\boxed{y(t) \to 0 \quad \text{implies} \quad y_n \to 0}$$

more interested in this implication... (so errors don't grow exponentially.)

Since exactly the same then

$$y(t) \to 0 \text{ if and only if } y_n \to 0.$$

A-stability:

Definition $\quad \mathbb{C}^- = \{ z \in \mathbb{C} : \operatorname{Re} z < 0 \}$

$$\mathcal{D} = \{ z \in \mathbb{C} : y_n \to 0 \quad \text{as} \quad n \to \infty \}$$

A method is A-stable if $\quad \mathbb{C}^- \subseteq \mathcal{D}$.

A-stability of RK methods...

Tableaux:
$$\begin{array}{c|c} c & A \\ \hline & b \end{array} \qquad\qquad c_j = \sum_{i=1}^{\nu} a_{ji}$$

$$\xi_j = y_n + h \boxed{\sum_{i=1}^{\nu} a_{ji}} f(t_n + c_i h, \xi_i)$$

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j)$$

Find the stability domain:

Approximate solution to

$$y' = \lambda y \qquad \text{where} \quad y(0) = 1$$

$$\xi_j = y_n + h \sum_{i=1}^{\nu} a_{ji} \lambda \xi_i$$

like matrix multiplication.

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\nu \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\nu} \\ \vdots & & & \\ a_{\nu 1} & a_{\nu 2} & \cdots & a_{\nu\nu} \end{bmatrix}$$

$$\xi = \begin{bmatrix} y_n \\ y_n \\ \vdots \\ y_n \end{bmatrix} + h\lambda A \xi = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} y_n + h\lambda A \xi$$

$$(I - h\lambda A)\xi = \mathbb{1} y_n \qquad \text{where } \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^\nu$$

$$\xi = (I - h\lambda A)^{-1} \mathbb{1} y_n$$

Also

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} \underbrace{b_j \lambda \xi_j}_{\text{dot product}} = y_n + h\lambda\, b \cdot \xi$$

where $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\nu \end{bmatrix}$ from the RK tableaux.

$$y_{n+1} = y_n + h\lambda b \cdot (I - h\lambda A)^{-1} \mathbb{1} y_n$$

$$= \left(1 + h\lambda b \cdot (I - h\lambda A)^{-1} \mathbb{1}\right) y_n$$

by induction

$$y_n = \left(1 + h\lambda b \cdot (I - h\lambda A)^{-1} \mathbb{1}\right)^n y_0$$

$$= \left(1 + h\lambda b \cdot \underbrace{(I - h\lambda A)^{-1}}_{\text{what's this?}} \mathbb{1}\right)^n$$

set $z = h\lambda$ note this substitution always gets rid of all the h's and λ's simultaneously.

need to find

$$\mathcal{D} = \left\{ z \in \mathbb{C} : \left| 1 + z\, b \cdot (I - zA)^{-1} \mathbb{1} \right| < 1 \right\}.$$

but what is $(I - zA)^{-1}$? Cramer's rule...

Cramers rule

$$C^{-1} = \frac{\operatorname{adj} C}{\det C}$$

we know what det C is but what was adj C ?

$$(I - zA)^{-1} = \frac{\operatorname{adj}(I - zA)}{\det(I - zA)}$$

What is $\det(I - zA)$?

Case that A corresponds to an explicit RK method...

$$\begin{array}{c|ccccc} & 0 & 0 & 0 & 0 & 0 \\ & ? & 0 & 0 & 0 & 0 \\ c & ? & ? & 0 & 0 & 0 \\ & ? & ? & ? & 0 & 0 \\ & ? & ? & ? & ? & 0 \\ \hline & & b & & & \end{array}$$

For explicit method A is lower triangular with 0's on the diagonal

Thus $I - zA$ is lower triangular with 1's on the diagonal

$$\det(I - zA) = \underbrace{1 \cdot 1 \cdot 1 \cdots 1}_{\nu \text{ terms.}} = 1$$

For ERK method

$$(I - zA)^{-1} = \mathrm{adj}(I - zA)$$

$$I - zA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -z? & 1 & 0 & 0 \\ -z? & -z? & 1 & 0 \\ -z? & -z? & -z? & 1 \end{bmatrix}$$

Minor

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -z? & 1 & 0 & 0 \\ -z? & -z? & 1 & 0 \\ -z? & -z? & -z? & 1 \end{bmatrix} = \det \begin{bmatrix} 0 & 0 & 0 \\ -z? & 1 & 0 \\ -z? & -z? & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -z? & 1 & 0 & 0 \\ -z? & -z? & 1 & 0 \\ -z? & -z? & -z? & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ -z? & 0 & 0 \\ -z? & -z? & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -z? & 1 & 0 & 0 \\ -z? & -z? & 1 & 0 \\ -z? & -z? & -z? & 1 \end{bmatrix} = \det \begin{bmatrix} -z? & 1 & 0 \\ -z? & -z? & 0 \\ -z? & -z? & 1 \end{bmatrix} \neq 0$$

has some powers of z in it

at most $\det\left((I - zA)_{ij\,\text{minor}} \right)$ is a polynomial of degree $\nu - 1$.

$$(I - zA)^{-1} = \mathrm{adj}(I - zA)$$

$$[\mathrm{adj}\, C]_{ij} = (-1)^{i+j} \det(C_{ji\text{-minor}})$$

so each entry in $\mathrm{adj}(I-zA)$ is at most a polynomial of degree $\nu - 1$.

one more z

$$y_n = \left(1 + z\, b \cdot \mathrm{adj}(I - zA)\, \mathbb{1}\right)^n$$

this product is just some weighted sum of sums of polynomials of degree $\nu-1$ and so it's also a polynomial of degree $\nu-1$

$y_n = (p(z))^n$ where p is a polynomial of degree ν

$$\mathcal{D} = \{ z \in \mathbb{C} \; : \; |p(z)| < 1 \}$$

Note that $|p(z)| \to \infty$ as $|z| \to \infty$

in particular is $z = a$ and $a \to -\infty$ then $|p(a)| > 1$ for a negative enough

Therefore $\mathbb{C}^- \not\subseteq \mathcal{D}$.

Conclusion: No explicit RK method has the property that $y_n \to 0$ as $n \to \infty$ for all values of h where $\mathrm{re}\, \lambda < 0$ (i.e. where it's supposed to).

Other case Implicit RK methods:

$$(I - zA)^{-1} = \frac{\operatorname{adj}(I - zA)}{\det(I - zA)}$$

no longer is $\det(I-zA) = 1$ but instead a matrix with lots of z's in it, maybe ν of them in a row.

$\det(I - zA)$ is a polynomial in z of at most degree ν.

Thus

$$(I - zA)^{-1} = \frac{\operatorname{adj}(I - zA)}{\det(I - zA)}$$

polynomial of degree at most $\nu - 1$

polynomial of degree at most ν

rational functions, quotients of polynomials...

Now use complex analysis to see whether

$$\left| 1 + z\, b \cdot \frac{\operatorname{adj}(I - zA)}{\det(I - zA)} \mathbb{1} \right| < 1 .$$

Next time... Lab...