$$\mathcal{D} = \left\{ z : \begin{array}{l} y_n \to 0 \text{ when } h\lambda = z \text{ and } y_n \text{ is} \\ \text{the approximation of } y' = \lambda y \text{ with } y(0) = 1 \end{array} \right\}$$

Euler's method.

$$y' = \lambda y \qquad f(t,y) = \lambda y$$

$$y_{n+1} = y_n + h f(t_n, y_n) = y_n + h\lambda y_n$$

$$= (1 + h\lambda) y_n = (1+z) y_n = r(z) y_n$$

where $r(z) = 1 + z$.

$$\mathcal{D} = \{ z : |r(z)| < 1 \}$$

Generalize: Any 1-step method where

$$y_{n+1} = r(\lambda y) y_n \qquad \text{when solving } y' = \lambda y$$

Then

$$\mathcal{D} = \{ z : |r(z)| < 1 \}$$

Whole family of 1-step methods is given by RK tableau

$$\begin{array}{c|c} b & A \\ \hline & c \end{array}$$

all of them look like

$$y_{n+1} = r(\lambda y) y_n \qquad \text{when solving } y' = \lambda y$$

Now use complex analysis to see whether

$$\left| 1 + \underbrace{z\, b \cdot \frac{\operatorname{adj}(I - zA)}{\det(I - zA)} \mathbb{1}}_{r(z)} \right| < 1.$$

$y_{n+1} = r(z)\, y_n$ when solving $y' = \lambda y$.

$\operatorname{adj}(I - zA)$ = matrix with polynomial entries in z of degree at most $\nu - 1$.

$\det(I - zA)$ = polynomial in z of degree at most ν.

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{\nu}$$

Thus $\underset{\text{matrix}}{\operatorname{adj}(I - zA)}\ \underset{\text{vector}}{\mathbb{1}}$ = vector with polynomial entries in z of degree at most $\nu - 1$.

$b \cdot \operatorname{adj}(I - zA) \mathbb{1}$ = polynomial in z of degree at most $\nu - 1$

just numbers...

$z\, b \cdot \operatorname{adj}(I - zA) \mathbb{1}$ = polynomial in z of degree at most ν

Thus

$$\frac{z\, b \cdot \operatorname{adj}(I - zA) \mathbb{1}}{\det(I - zA)}$$ = rational function with numerator and denominators both polynomials of degree at most ν.

$$r(z) = 1 - \frac{z\, b \cdot \operatorname{adj}(I - zA)\mathbb{1}}{\det(I - zA)}$$

also a rational function with numerator and denominators both polynomials of degree at most s.

Recall

$\mathbb{C}^- = \{ z : \operatorname{Re} z < 0 \}$ left half of the complex plane...

$\mathcal{D} = \{ z : |r(z)| < 1 \}$

Question is $\mathbb{C}^- \subseteq \mathcal{D}$? If so then A-stable...

- So everywhere the solution was supposed to tend to zero as $t \to \infty$ the approximation of that solution actually does.

Use complex analysis to understand this...

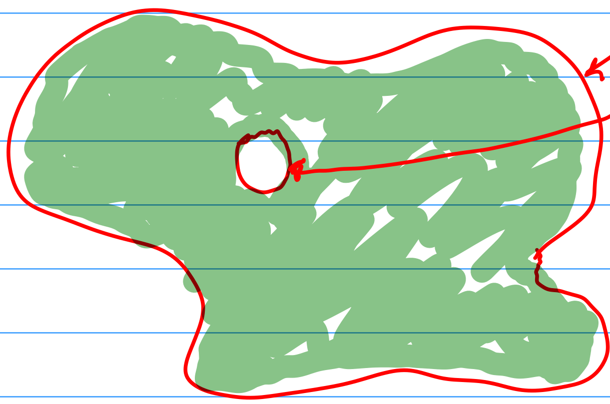
Theorem: if $r(z)$ is analytic on a domain Ω then

set with a nice boundary..

Max value principle

$$\max\{ |r(z)| : z \in \Omega \} = \max\{ |r(z)| : z \in \partial\Omega \}$$

boundary of Ω



$\partial\Omega$ the boundary is in red.

Analytic means differentiable as a function with a complex argument.. That is

$$\lim_{h\to 0} \frac{r(z+h)-r(z)}{h}$$

exists where $h\in\mathbb{C}$ tending to 0,

In particular

$$\lim_{h\to 0} \frac{r(z+ih)-r(z)}{ih} \quad \text{exists}$$

when $h\in\mathbb{R}$ tending to zero and also..,

$$\lim_{h\to 0} \frac{r(z+h)-r(z)}{h} \quad \text{exists}$$

when $h\in\mathbb{R}$ tending to zero

Note that $h\to 0$ through complex values is a lot of ways to go to zero, so the existence of a limit in this case is a much stronger condition...

Thus a complex function being differentiable is a very strong hypothesis and one can deduce lots of consequences... In particular...

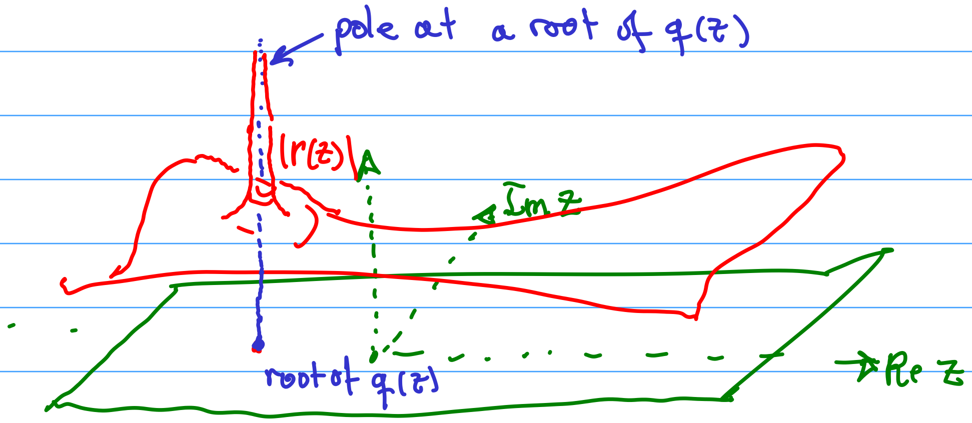
$$\max\{|r(z)| : z\in\Omega\} = \max\{|r(z)| : z\in\partial\Omega\}$$

How can I use this ....

$$r(z) = \frac{p(z)}{q(z)}$$ where p and q are polynomials in z of degree ν.

Note that polynomials are differentiable even in the complex sense... power rule is the same as before...

The roots where $q(z)=0$ are called the poles or $r(z)$.
asymptotes in the complex plane...



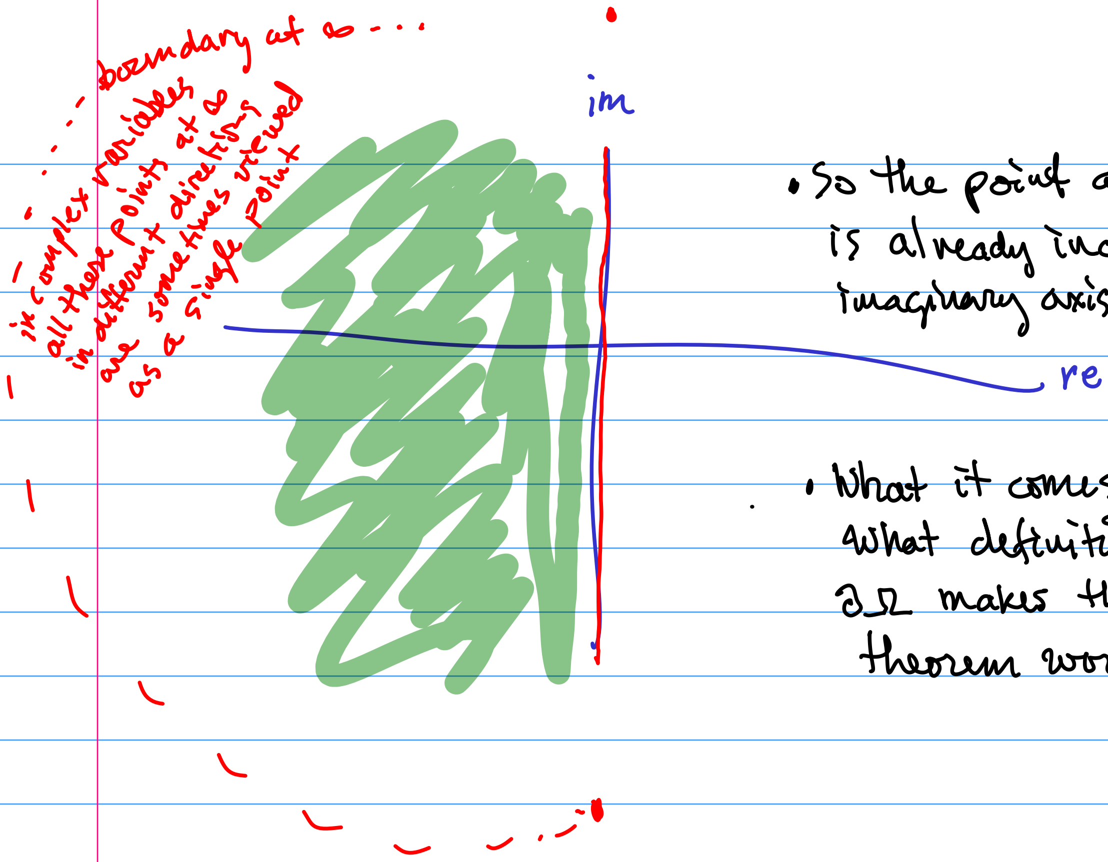
If there are any poles in $\mathbb{C}^-$ then its not posible $|r(z)|<1$ in all of $\mathbb{C}^-$ because it actually reaches ∞ at the poles... Thus, the roots of q must be in the right hand plane if the method is A-stable.

If all the poles are in the right hand plane, it means that $r(z)$ is differentiable everywhere in the left plane... which means the maximum principle holds with $\Omega = \mathbb{C}^-$.

$$\max\{|r(z)| : z \in \mathbb{C}^-\} = \max\{|r(z)| : z \in \partial\mathbb{C}^-\}$$

imaginary axis...

$$= \max\{|r(it)| : t \in \mathbb{R}\}$$



- So the point at infinity is already included on the imaginary axis...

- What it comes down to is what definition of 'boundary $\partial\Omega$' makes the proof of the theorem work.

We thus have...

Our first observation is that there is no need to check every $z \in \mathbb{C}^-$ to verify that a given rational function r originates in an A-stable method (such an r is called *A-acceptable*).

Lemma 4.3 *Let r be an arbitrary rational function that is not a constant. Then $|r(z)| < 1$ for all $z \in \mathbb{C}^-$ if and only if all the poles of r have positive real parts and $|r(it)| \le 1$ for all $t \in \mathbb{R}$.*

Finishes Chapter 4 unless we do multistep method stability...

Next thing is PDE's ... skip to chapter 8.

Before PDE's a little bit about implicit methods...

Simplest example: Trapezoid method

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

This is an A-stable method of order 2.

Other colocation methods from Gauss Quadrature.

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

<u>Note</u>: all colocation methods are implicit... also NO explicit RK method is A-stable...

How to actually use an implicit method?

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1},y_{n+1})\right)$$

Solve for z such that

$$z = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1}, z)\right)$$

and then set $y_{n+1} = z$.

Seems like I need to invert the function f: Write this as a root finding problem

$$\varphi(z) = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1}, z)\right) - z$$

Use some numerical technique to find z such that

$$\varphi(z) = 0.$$

In the case f is linear (i.e. $f(z) = Ax$) then this is easy to solve... but what if f is nonlinear.

$$y' = y^2\cos(t) \qquad y(0) = 0.8$$

in this case maybe the quadratic formula... but what about in general?

In general we find z iteratively using better and better approximations...

A-stability of the trapezoid method

stability of the root finding method...

To find z such that $\varphi(z)=0$ two ideas

$$\varphi(z) = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_{n+1}, z)\right) - z$$

(i) Newtons method.

$$g(z) = z - \frac{\varphi(z)}{\varphi'(z)}$$

$z_0 =$ initial guess.

$z_{n+1} = g(z_n)$ take limit z_n as $n\to\infty$
which due to quadratic convergence of Newton's method may only take one or two iterations depending of z_0.

Initial guess

$z_0 = y_n$ since y_{n+1} should be close to y_n

$z_0 = y_n + hf(t_n,y_n)$ might be closer to y_{n+1}

Use an explicit method to initialize the guess for Newton's method... The initial guess could also be obtained by bisections ... the idea is to preserve the A-stability of the trapezoid method...

we'll revisit this next week after the exam

$$y' = y^2 \cos(t) \qquad y(0) = 0.8$$

For next week find the exact solution of this ODE.