We're done with ODE's

Skip chapters 5 and 6 which go into more depth for certain ODE techniques.

Now in chapter 8

8

Finite difference schemes

bi-infinite sequence (for now)

Setup: $z_k = z(kh)$ where $h > 0$ for $k \in \mathbb{Z}$

Samples of z (grid points)

on the computer we have finite sequences and boundary conditions

Define:

$\mathcal{E} z_k = z_{k+1}$ shift operator on the sequence

$I z_k = z_k$

$\mathcal{E}(\{z_k\}_{k\in\mathbb{Z}}) = \{z_{k+1}\}_{k\in\mathbb{Z}}$

Shorthand for this

these three are small

$\Delta_+ z_k = z_{k+1} - z_k$ forward difference operator

$\Delta_- z_k = z_k - z_{k-1}$ backwards difference operator

$\Delta_0 z_k = \underbrace{z_{k+\frac{1}{2}}} - \underbrace{z_{k-\frac{1}{2}}}$ central difference operator

these aren't integers so not really grid points

recall

$z_k = z(kh)$

so

$z_{k+\frac{1}{2}} = z(kh + \frac{1}{2}h)$

$\Upsilon_0 z_k = \dfrac{z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}}{2}$ averaging operation

What do I mean that $\Delta_+ z_k$ is small?

If z is differentiable then

$$|\Delta_+ z_k| = |z_{k+1} - z_k| = |z(kh+h) - z(kh)|$$

$$= \left| \int_{kh}^{kh+h} z'(s)\,ds \right| \leq \int_{kh}^{kh+h} |z'(s)|\,ds$$

take max of $|z'|$ out

$$\leq \max\{|z'(s)| : s \in [kh, kh+h]\} \int_{kh}^{kh+h} 1\,ds$$

$$= \max\{|z'(s)| : s \in [kh, kh+h]\}\, h$$

generally if z' is continuous this maximum exists and if we let $B = \max\{|z'(s)| : s \in [-1,1]\}$ then

$$|\Delta_+ z_k| \leq B h \qquad \text{for } h < \max\left\{\frac{1}{|k|}, \frac{1}{|k+1|}\right\}$$

In other words

$$|\Delta_+ z_k| = O(h) \qquad \text{as } h \to 0.$$

Recall:

$$E z_k = z_{k+1} \qquad \Delta_+ z_k = z_{k+1} - z_k \qquad \Delta_0 z_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$$

$$I z_k = z_k \qquad \Delta_- z_k = z_k - z_{k-1} \qquad \gamma_0 z_k = \frac{z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}}{2}$$

Note they are linear...

$z_k = z(hk)$

$x_k = x(hk)$

$w_k = (\alpha z + \beta x)(hk)$

Thus $w_k = \alpha z_k + \beta x_k$.

$$\Delta_- w_k = w_k - w_{k-1} = \alpha z_k + \beta x_k - (\alpha z_{k-1} + \beta x_{k-1})$$

$$= \alpha(z_k - z_{k-1}) + \beta(x_k - x_{k-1}) = \alpha \Delta_- z_k + \beta \Delta_- x_k$$

Compositions of these difference operators

$$\mathcal{E} z_k = z_{k+1} \qquad \Delta_+ z_k = z_{k+1} - z_k \qquad \Delta_0 z_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$$

$$I z_k = z_k \qquad \Delta_- z_k = z_k - z_{k-1} \qquad \Upsilon_0 z_k = \frac{z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}}{2}$$

Example:

$$\Delta_+^2 z_k = \Delta_+(\Delta_+ z_k) = \Delta_+(z_{k+1} - z_k) = \Delta_+ z_{k+1} - \Delta_+ z_k$$

$$= z_{k+2} - z_{k+1} - (z_{k+1} - z_k) = z_{k+2} - 2z_{k+1} + z_k$$

Example:

$$\Upsilon_0 \Delta_0 z_k = \Upsilon_0 z_{k+\frac{1}{2}} - \Upsilon_0 z_{k-\frac{1}{2}}$$

these operators involve points off the grid...

$$= \frac{z_{k+1} + z_k}{2} - \frac{z_k + z_{k-1}}{2} = \frac{z_{k+1} - z_{k-1}}{2}$$

note these points are on the grid since $k+1$ and $k-1$ are integers.

If we only knew z_k's on the grid, how could we figure out $z_{k+\frac{1}{2}}$ and so forth? Interpolation...

Functional calculus of finite differences...

But first linear algebra... the same idea..

since the finite difference operators are linear, this is natural analogy..

$A \in \mathbb{R}^{n\times n}$ then A^2 makes sense

$\alpha A + \beta A$ makes sense

$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

also makes sense...

Taylor's theorem allows one to write lots of functions as the limit of a polynomial...

$$e^x = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} x^j = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$$

Thus

$$e^{A} = \sum_{j=0}^{\infty} \frac{1}{j!} A^{j}$$

makes sense (if it converges). It does because no matter how large $\|A\|$ is the $j!$ in the denominator gets bigger..

For finite differences can define the same thing

$$e^{\Delta_+} = \sum_{j=0}^{\infty} \frac{1}{j!} (\Delta_+)^{j}$$

on the sequences

$$(e^{\Delta_+}) u_k = \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_+^{j} u_k$$

What other functions can be written as the limit of polynomials? Analytic function by definition...

Example: Newton's binomial theorem.

$$(1+x)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^{j} \quad \text{for} \quad |x|<1.$$

where

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-j+1)}{1\cdot 2\cdot 3\cdot 4\cdots j}$$

Plug in a finite difference operation. Since x needs to be small I'll plug in Δ_+. Thus,

$$(I+\Delta_+)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} (\Delta_+)^{j}$$

$\Delta_+ z_k = z_{k+1} - z_k$

Note

$$(I+\Delta_+) z_k = z_k + z_{k+1} - z_k = z_{k+1} = E z_k$$

Thus,

$$\mathcal{E}^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} (\Delta_+)^j$$

Specifically $\alpha = \frac{1}{2}$

$$\sqrt{\mathcal{E}} = \sum_{j=0}^{\infty} \binom{1/2}{j} (\Delta_+)^j$$

analytical expression for $\sqrt{\mathcal{E}}$

Algebraic meaning of $\sqrt{\mathcal{E}}$ is any operator A such that

$$\mathcal{E} = A^2 \qquad \text{then} \qquad \sqrt{\mathcal{E}} = A.$$

$$\mathcal{E} z_k = z_{k+1} = z(x+h) \qquad \text{where} \qquad x = kh$$

$$= z(x + \tfrac{1}{2}h + \tfrac{1}{2}h) = A^2 z_k \qquad \text{where} \qquad A z_k = z_{k+\frac{1}{2}}$$

Thus

$$\sqrt{\mathcal{E}}\, z_k = z_{k+\frac{1}{2}}$$

$$\sqrt{\mathcal{E}}^{\,2} z_k = \sqrt{\mathcal{E}}\sqrt{\mathcal{E}}\, z_k = \sqrt{\mathcal{E}}\, z_{k+\frac{1}{2}} = z_{k+1} = \mathcal{E} z_k$$

Therefor

$$\sqrt{\mathcal{E}}\, z_k = \sum_{j=0}^{\infty} \binom{1/2}{j} (\Delta_+)^j z_k$$

$$z_{k+\frac{1}{2}} = \sum_{j=0}^{\infty} \binom{1/2}{j} (\Delta_+)^j z_k$$

approximate

$$z_{k+\frac{1}{2}} \approx \sum_{j=0}^{n-1} \binom{1/2}{j} (\Delta_+)^j z_k$$

This is an interpolation

How accurate, Since $\Delta_+ z_k = O(h)$

$$z_{k+\frac{1}{2}} = \sum_{j=0}^{n-1} \binom{1/2}{j} (\Delta_+)^j z_k + O(h^n)$$

Interested in differential equations... Think about derivatives...

Setup: $z_k = z(kh)$ where $h > 0$ for $k \in \mathbb{Z}$

Define a differential operator

$$Dz_k = z'(kh)$$

Taylor series

$$z(x+h) = z(x) + h z'(x) + \frac{h^2}{2!} z''(x) + \frac{h^3}{3!} z^{(3)}(x) + \cdots$$

$$z(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} z^{(j)}(x)$$

Set $x = kh$ then

$$z_{k+1} = \sum_{j=1}^{\infty} \frac{h^j}{j!} D^j z_k = \boxed{\left(\sum_{j=1}^{\infty} \frac{1}{j!} (hD)^j \right)} z_k$$

compare with

$$e^x = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$$

Therefore

$$\mathcal{E} z_k = e^{hD} z_k \quad \text{or} \quad \mathcal{E} = e^{hD}$$

take logarithms (formally)

$$\ln \mathcal{E} = hD$$

$$D = \frac{1}{h} \ln \mathcal{E}$$

derivatives

in terms of finite difference operator

$z'(hk) = Dz_k = \frac{1}{h}(\ln \mathcal{E}) z_k$ but what is $\ln \mathcal{E}$?

recall

$$(I+\Delta_+)z_k = z_k + z_{k+1} - z_k = z_{k+1} = \mathcal{E} z_k$$

$$z'(hk) = \frac{1}{h}\ln(I+\Delta_+)\, z_k$$

here $\Delta_+ = O(h)$ is small..

Taylor series for logarithm

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

not factorials in the denominator
so only converges when $|x|<1$.

Thus

$$\ln(I+\Delta_+) = \Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3 - \frac{1}{4}\Delta_+^4 + \dots$$

$$\ln(I+\Delta_+) = \Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3 + O(h^4)$$

It follows that

$$z'(hk) = \frac{1}{h}\ln(I+\Delta_+)\, z_k$$

$$= \frac{1}{h}\left(\Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3\right) z_k + O(h^4)$$

or

$$z'(hk) \approx \frac{1}{h}\left(\Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3\right) z_k$$

Basically the notation

$\mathcal{E} z_k = z_{k+1}$ $\quad$ $\Delta_+ z_k = z_{k+1} - z_k$ $\quad$ $\Delta_0 z_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$

$I z_k = z_k$ $\quad$ $\Delta_- z_k = z_k - z_{k-1}$ $\quad$ $\gamma_0 z_k = \dfrac{z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}}{2}$

is nice enough that we can finish complicated stuff.