$$(I+\Delta_+)z_k = z_k + z_{k+1} - z_k = z_{k+1} = \mathcal{E} z_k$$

Thus,

$$\mathcal{E} = I + \Delta_+ \qquad \mathcal{E} = e^{hD}$$

Result in this approximation.

$$z'(hk) \approx \frac{1}{h}\left(\Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3\right) z_k$$

Next time is in class computing lab...

Apr 24-Apr 28 Week 13: 8.2 Two-point boundary problems (Lab 5)

Recall that

$$\mathcal{E} z_k = z_{k+1} \qquad \Delta_+ z_k = z_{k+1} - z_k \qquad \Delta_0 z_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$$

$$I z_k = z_k \qquad \Delta_- z_k = z_k - z_{k-1} \qquad \gamma_0 z_k = \frac{z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}}{2}$$

is nice enough that we can finish complicated stuff.

Put Δ_0 back on the grid...

$$\Delta_0 z_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} \qquad \mathcal{E}^{1/2} u_k = u_{k+1/2}$$

$$\mathcal{E}^{1/2}\Delta_0 z_k = z_{k+1} - z_k$$

same

Remark

$$\Delta_0 \mathcal{E}^{1/2} z_k = \Delta_0 z_{k+1/2} = z_{k+\frac{1}{2}+\frac{1}{2}} - z_{k+\frac{1}{2}-\frac{1}{2}} = z_{k+1} - z_k$$

$$\Delta_0 \mathcal{E}^{1/2} z_k = \mathcal{E} z_k - z_k = \left((\mathcal{E}^{1/2})^2 - I\right) z_k$$

Therefore

$$\Delta_0 \mathcal{E}^{1/2} = (\mathcal{E}^{1/2})^2 - I$$

$$\underbrace{(\mathcal{E}^{1/2})^2 - \Delta_0 \mathcal{E}^{1/2} - I = 0}_{\text{quadratic equation in } \mathcal{E}^{1/2}}$$

recall, we know know how to take square roots of difference operators, so we can solve quadratic equations involving finite difference operators...

Solve for $\mathcal{E}^{1/2}$ using quadratic formula...

$a = I \quad , \quad b = -\Delta_0 \quad , \quad c = -I$

$$\mathcal{E}^{1/2} = (2a)^{-1}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

$$= (2I)^{-1}\left(\Delta_0 \pm \sqrt{\Delta_0^2 + 4I}\right)$$

$$= \frac{1}{2}\left(\Delta_0 \pm \sqrt{\Delta_0^2 + 4I}\right)$$

which is correct + or − ?
Take $h \to 0$ to check.

$$\mathcal{E} u_k = u_{k+1} = u(x+h) \to u(x) \text{ as } h \to 0$$

holding x fixed

$\mathcal{E} \to I$ as $h \to 0$

$\Delta_0 = O(h) \to 0$ as $h \to 0$

$$\mathcal{E}^{1/2} = \frac{1}{2}\left(\Delta_0 \pm \sqrt{\Delta_0^2 + 4I}\right)$$

$$I \qquad \frac{1}{2}\left(0 \pm \sqrt{0^2 + 4I}\right)$$

$$\pm \frac{1}{2}\sqrt{4I} = \pm\sqrt{I} =: \pm I$$

Thus $$\boxed{\mathcal{E}^{1/2} = \frac{1}{2}\left(\Delta_0 + \sqrt{\Delta_0^2 + 4I}\right)}$$

$$\mathcal{E} = \left(\frac{1}{2}\left(\Delta_0 + \sqrt{\Delta_0^2 + 4I}\right)\right)^2$$

$$\mathcal{E} = e^{hD} \qquad hD = \log \mathcal{E}$$

$$D = \frac{1}{h}\log \mathcal{E} = \frac{1}{h}\log\left(\frac{1}{2}\left(\Delta_0 + \sqrt{\Delta_0^2 + 4I}\right)\right)^2$$

$$= \frac{2}{h}\log\left(\frac{1}{2}\left(\Delta_0 + \sqrt{\Delta_0^2 + 4I}\right)\right)$$

Two reasons for doing this.

- Get comfortable with notation
- Some formulas we obtain are useful.

So what's this?

$$\frac{2}{h}\log\left(\tfrac{1}{2}\left(\Delta_0+\sqrt{\Delta_0^2+4I}\right)\right) \qquad \xi=\tfrac{1}{2}\Delta_0 \quad \Delta_0=2\xi$$

$$g(\xi)=\log\left(\xi+\tfrac{1}{2}\sqrt{4\xi^2+4}\right)=\log\left(\xi+\sqrt{\xi^2+1}\right)$$

$$g'(\xi)=\frac{1}{\xi+\sqrt{\xi^2+1}}\,\frac{d}{d\xi}\left(\xi+\sqrt{\xi^2+1}\right)$$

$$=\frac{1}{\xi+\sqrt{\xi^2+1}}\left(1+\frac{\xi}{\sqrt{\xi^2+1}}\right).$$

$$=\frac{1}{\xi+\sqrt{\xi^2+1}}\left(\frac{\sqrt{\xi^2+1}+\xi}{\sqrt{\xi^2+1}}\right)=\frac{1}{\sqrt{\xi^2+1}}$$

$$=(\xi^2+1)^{-1/2}$$

$\xi^2=\frac{1}{4}\Delta_0^2=O(h^2)$ and this is small...

Binomial theorem to find the Taylor series

Example: Newton's binomial theorem.

$$(1+x)^\alpha=\sum_{j=0}^{\infty}\binom{\alpha}{j}x^j \quad \text{for} \quad |x|<1.$$

Thus

$$(1+\xi^2)^{-1/2}=\sum_{j=0}^{\infty}\binom{-1/2}{j}\xi^{2j}$$

$g(0) = \log(0 + \sqrt{0^2+1})$
$= \log 1 = 0$

$$g(\xi) = \log(\xi + \sqrt{\xi^2 + 1})$$

$$g'(\xi) = (1+\xi^2)^{1/2} = \sum_{j=0}^{\infty} \binom{-1/2}{j} \xi^{2j} \quad \text{for } |\xi| < 1$$

$$D = \frac{2}{h} \log\left(\tfrac{1}{2}(\Delta_0 + \sqrt{\Delta_0^2 + 4I})\right) = \frac{2}{h} g\left(\tfrac{1}{2}\Delta_0\right)$$

Thus,

$$g(\xi) - g(0) = \int_0^{\xi} g'(\tau)\, d\tau$$

$$= \int_0^{\xi} \sum_{j=0}^{\infty} \binom{-1/2}{j} \tau^{2j}\, d\tau$$

under the assumption $|\xi| < \alpha$ where $\alpha \in (0,1)$ the series converges uniformly

$$= \sum_{j=0}^{\infty} \binom{-1/2}{j} \int_0^{\xi} \tau^{2j}\, d\tau$$

$$= \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \xi^{2j+1}$$

$g(0) = \log(0 + \sqrt{0^2+1})$
$= \log 1 = 0$

So

$$g(\xi) = \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \xi^{2j+1}$$

odd powers of Δ_0 are not on the grid...

$$Du_k = \frac{2}{h} g\left(\tfrac{1}{2}\Delta_0\right) = \frac{2}{h} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j+1} u_k$$

For this to be useful we want the terms on the right to be on the grid...

One idea factor Δ_0 out to get even powers...

Before that note

$$D^2 u_k = \left(\frac{2}{h} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j+1} \right)^2 u_k$$

$$= \frac{4}{h^2} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j+1} \sum_{l=0}^{\infty} \frac{1}{2l+1} \binom{-1/2}{l} \left(\tfrac{1}{2}\Delta_0\right)^{2l+1}$$

$$= \frac{4}{h^2} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \frac{1}{2l+1} \binom{-1/2}{l} \left(\tfrac{1}{2}\Delta_0\right)^{2j+2l+2}$$

even powers of Δ_0 so this on the grid...

Back to the idea of factoring out Δ_0

$$D u_k = \frac{2}{h} g\left(\tfrac{1}{2}\Delta_0\right) = \frac{2}{h} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j+1} u_k$$

$$= \Delta_0 \frac{1}{h} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j} u_k$$

this still puts it off the grid...

even powers on the grid

Technique related to $\gamma_0 u_k = \frac{1}{2}(u_{k+\frac{1}{2}} + u_{k-\frac{1}{2}})$.

$$\Delta_0 u_k = u_{k-\frac{1}{2}} - u_{k-\frac{1}{2}}$$

subtract

$$\gamma_0^2 u_k = \gamma_0 \, \tfrac{1}{2}\left(u_{k+\frac{1}{2}} + u_{k-\frac{1}{2}}\right) = \tfrac{1}{4}\left(u_{k+1} + 2u_k + u_{k-1}\right)$$

$$\tfrac{1}{4}\Delta_0^2 u_k = \tfrac{1}{4}\Delta_0\left(u_{k+\frac{1}{2}} - u_{k-\frac{1}{2}}\right) = \tfrac{1}{4}\left(u_{k+1} - 2u_k + u_{k-1}\right)$$

$$\left(\gamma_0^2 - \tfrac{1}{4}\Delta_0^2\right)u_k = u_k = I u_k$$

Thus

$$\gamma_0^2 - \tfrac{1}{4}\Delta_0^2 = I$$

$$\gamma_0 = \sqrt{I + \tfrac{1}{4}\Delta_0^2}$$

$$h(\xi) = (1+\xi^2)^{-1/2}$$

and

$$\gamma_0^{-1} = \left(I + \tfrac{1}{4}\Delta_0^2\right)^{-1/2} = h\left(\tfrac{1}{2}\Delta_0\right)$$

$$h(\xi) = \sum_{j=0}^{\infty} \binom{-1/2}{j} \xi^{2j}$$

...same as $g'(\xi)$ from before

$$\gamma_0^{-1} = h\left(\tfrac{1}{2}\Delta_0\right) = \sum_{j=0}^{\infty} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j}$$

Therefore

$$I = \gamma_0 \gamma^{-1} = \boxed{\gamma_0 \sum_{j=0}^{\infty} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j}}$$

$$Du_k = \Delta_0 \, I \, \frac{1}{h} \sum_{j=0}^{\infty} \frac{1}{2j+1} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j} u_k$$

$$= \Delta_0 \gamma_0 \sum_{j=0}^{\infty} \binom{-1/2}{j} \left(\tfrac{1}{2}\Delta_0\right)^{2j} \frac{1}{h} \sum_{l=0}^{\infty} \frac{1}{2l+1} \binom{-1/2}{l} \left(\tfrac{1}{2}\Delta_0\right)^{2l} u_k$$

$$= \frac{1}{h} \underbrace{\Delta_0 \gamma_0}_{\text{also on the grid.}} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \binom{-1/2}{j} \frac{1}{2l+1} \binom{-1/2}{l} \left(\tfrac{1}{2}\Delta_0\right)^{2j+2l}$$

on the grid.

Question is $\gamma_0 \Delta_0 = \Delta_0 \gamma_0$?

Example:

$$\gamma_0 \Delta_0 z_k = \gamma_0 z_{k+\frac{1}{2}} - \gamma_0 z_{k-\frac{1}{2}}$$

these operators involve points

$$= \frac{z_{k+1} + z_k}{2} - \frac{z_k + z_{k-1}}{2} = \frac{z_{k+1} - z_{k-1}}{2}$$

note these on the

$$\Delta_0 \gamma_0 z_k = \Delta_0 \tfrac{1}{2}(z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}) = \tfrac{1}{2}\Delta_0 z_{k+\frac{1}{2}} + \tfrac{1}{2}\Delta_0 z_{k-\frac{1}{2}}$$

$$= \tfrac{1}{2}(z_{k+1} - z_k) + \tfrac{1}{2}(z_k - z_{k-1})$$

$$= \tfrac{1}{2}(z_{k+1} - z_{k-1})$$

$$D u_k = -\frac{1}{h}\Delta_0 \gamma_0 \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{-1/2}{j} \frac{1}{2l+1} \binom{-1/2}{l} \left(\tfrac{1}{2}\Delta_0\right)^{2j+2l}$$

This is even a useful formula... how to use?
decide what order of aproximation you want

Say $O(h^p)$ then use that $\Delta_0 = O(h)$ and
throw out the higher order terms from the sums.

Next time Chapter 8.2 solving Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

using finite differences ... in the lab ..