Want to show A has no zero eigenvalues... I.e. $0 \notin \sigma(A)$.

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julia> A
9×9 SparseMatrixCSC{Float64, Int64} with 33 stored entries:
1  -4.0   1.0    ⋅    1.0    ⋅     ⋅     ⋅     ⋅     ⋅
2   1.0  -4.0   1.0    ⋅    1.0    ⋅     ⋅     ⋅     ⋅
3    ⋅    1.0  -4.0    ⋅     ⋅    1.0    ⋅     ⋅     ⋅
4   1.0    ⋅     ⋅   -4.0   1.0    ⋅    1.0    ⋅     ⋅
5    ⋅    1.0    ⋅    1.0  -4.0   1.0    ⋅    1.0    ⋅
6    ⋅     ⋅    1.0    ⋅    1.0  -4.0    ⋅     ⋅    1.0
7    ⋅     ⋅     ⋅    1.0    ⋅     ⋅   -4.0   1.0    ⋅
8    ⋅     ⋅     ⋅     ⋅    1.0    ⋅    1.0  -4.0   1.0
9    ⋅     ⋅     ⋅     ⋅     ⋅    1.0    ⋅    1.0  -4.0

julia>
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The diagonal is −4 all along and there are some 1's off diagonal about four 1's per row except some missing at the boundary.

Why is this matrix invertible?

Answer because of the 4's are bigger than the 1's so the matrix is diagonally dominant..

Intuitively, If I throw out the smaller numbers only a diagonal matrix is left and that's invertible.

Lemma 8.3 (The Geršgorin criterion) *Let $B = (b_{k,\ell})$ be an arbitrary irreducible* (A.1.2.5) *complex $d \times d$ matrix. Then*

$$\sigma(B) \subset \bigcup_{i=1}^{d} \mathbb{S}_i,$$

where

disks

center of disk

radius of disk

$$\mathbb{S}_i = \left\{ z \in \mathbb{C} : |z - b_{i,i}| \le \overbrace{\sum_{j=1,\, j\neq i}^{d} |b_{i,j}|} \right\}$$

and $\sigma(B)$ is the set containing the eigenvalues of B. Moreover, $\lambda \in \sigma(B)$ may lie on $\partial\mathbb{S}_{i^0}$ for some $i^0 \in \{1, 2, \dots, d\}$ only if $\lambda \in \partial\mathbb{S}_i$ for all $i = 1, 2, \dots, d$. The $\mathbb{S}_i$ are known as Geršgorin discs.

Condition about invertibility: The matrix B is invertible is equivalent to saying all eigenvalues of B are non-zero.

Explanation: Exercise 8.8

8.8 In this exercise we prove, step by step, the Geršgorin criterion, which was stated in Lemma 8.3.

a Let $C = (c_{i,j})$ be an arbitrary $d \times d$ singular complex matrix. Then there exists $\boldsymbol{x} \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ such that $C\boldsymbol{x} = \mathbf{0}$. Choose $\ell \in \{1, 2, \ldots, d\}$ such that

$$|x_\ell| = \max_{j=1,2,\ldots,d} |x_j| > 0.$$

By considering the ℓth row of $C\boldsymbol{x}$, prove that

$$|c_{\ell,\ell}| \leq \sum_{j=1,\, j\neq \ell}^{d} |c_{\ell,j}|. \tag{8.35}$$

b Let B be a $d \times d$ matrix and choose $\lambda \in \sigma(B)$, where $\sigma(B)$ is the set containing the eigenvalues of B. Substituting $C = B - \lambda I$ in (8.35) prove that $\lambda \in \mathbb{S}_\ell$ (the Geršgorin discs $\mathbb{S}_i$ were defined in Lemma 8.3). Hence deduce that

$$\sigma(B) \subset \bigcup_{i=1}^{d} \mathbb{S}_i.$$

Given any matrix $B \in \mathbb{C}^{d\times d}$ with eigenvalue λ with eigenvector x
$C = B - \lambda I$ has a zero eigenvalue same eigenvector.

Thus,

$$Bx = \lambda x \quad \text{or} \quad \boxed{Cx = 0}$$

Since $x \neq 0$ (otherwise not an eigenvector) there is an entry of x with maximal magnitude.

Call that entry x_ℓ for some ℓ.

$$|x_i| \leq |x_\ell| \quad \text{for all } i = 1, \ldots, d$$

Thus, by definition of matrix multiplication

$$0 = (Cx)_i = \sum_{j=1}^{d} c_{ij} x_j = \sum_{j\neq \ell} c_{ij} x_j + c_{i\ell} x_\ell$$

Thus

$$c_{i\ell} x_\ell = -\sum_{j\neq\ell} c_{ij} x_j \qquad \text{this holds for } i = 1, 2, \ldots, d.$$

In particular taking $i = \ell$ yields

$$c_{\ell\ell} x_\ell = - \sum_{j \neq \ell} c_{\ell j} x_j$$

or

$$c_{\ell\ell} = - \sum_{j \neq \ell} c_{\ell j} \frac{x_j}{x_\ell} .$$

By the triangle inequality

$$|c_{\ell\ell}| \leq \sum_{j \neq \ell} |c_{\ell j}| \left| \frac{x_j}{x_\ell} \right|$$

recall this

$|x_i| \leq |x_\ell|$ for all $i = 1, \dots, d$

thus $\frac{|x_j|}{|x_\ell|} \leq 1$ for all $j = 1, \dots, d$

Consequently

$$|c_{\ell\ell}| \leq \sum_{j \neq \ell} |c_{\ell j}|$$

$$|c_{\ell,\ell}| \leq \sum_{j=1,\, j\neq \ell}^{d} |c_{\ell,j}|.$$

done with first part of exercise

Next part (b) reinterpret this in terms of the matrix B.

$C = B - \lambda I$ so $c_{\ell j} = \begin{cases} B_{\ell\ell} - \lambda & \text{if } \ell = j \\ B_{\ell j} & \text{if } \ell \neq j \end{cases}$

Plug it in to obtain...

$$|B_{\ell\ell} - \lambda| \leq \underbrace{\sum_{j \neq \ell} |B_{\ell j}|}_{\text{radius}}$$

eigenvalue is no further away from $B_{\ell\ell}$ than that radius.

In particular

$$\lambda \in \left\{ z \in \mathbb{C} : |B_{\ell\ell} - z| \le \sum_{j \neq \ell} |B_{\ell j}| \right\}.$$

$i=1$

disks $\downarrow$ center of disk $\swarrow$ radius

$$\mathbb{S}_i = \left\{ z \in \mathbb{C} : |z - b_{i,i}| \le \underbrace{\sum_{j=1,\, j\neq i}^{d} |b_{i,j}|}_{} \right\}$$

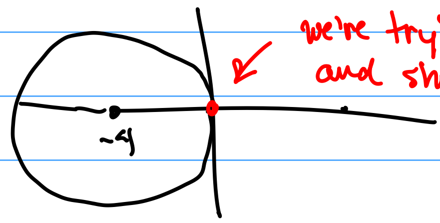
finish part (b) of Exercise and proof of theorem.

and $\sigma(B)$ is the set containing the eigenvalues of B. Moreover, $\lambda \in \sigma(B)$ may lie on $\partial\mathbb{S}_{i^0}$ for some $i^0 \in \{1, 2, \ldots, d\}$ only if $\lambda \in \partial\mathbb{S}_i$ for all $i = 1, 2, \ldots, d$. The $\mathbb{S}_i$ are known as Geršgorin discs.

- If an eigenvalue is in the boundary of one disk, then its in the boundary of all the disks.

note that $0 \in \partial S_5 = \partial \{ z \notin \mathbb{C} : |-4-z| \le 1+1+1+1 \}$

	1.0	·	·	-4.0	1.0	·	1.0	·	·
5	·	1.0	·	1.0	-4.0	1.0	·	1.0	·
	·	·	1.0	·	1.0	-4.0	·	·	1.0



we're trying to rule out 0 and show $0 \notin \sigma(A)$.

But if $\lambda = 0$ is an eigenvalue, it has to be on the boundary of all the disks..

7	·	·	·	1.0	·	·	-4.0	1.0	·
8	·	·	·	·	1.0	·	1.0	-4.0	1.0
9	·	·	·	·	·	1.0	·	1.0	-4.0

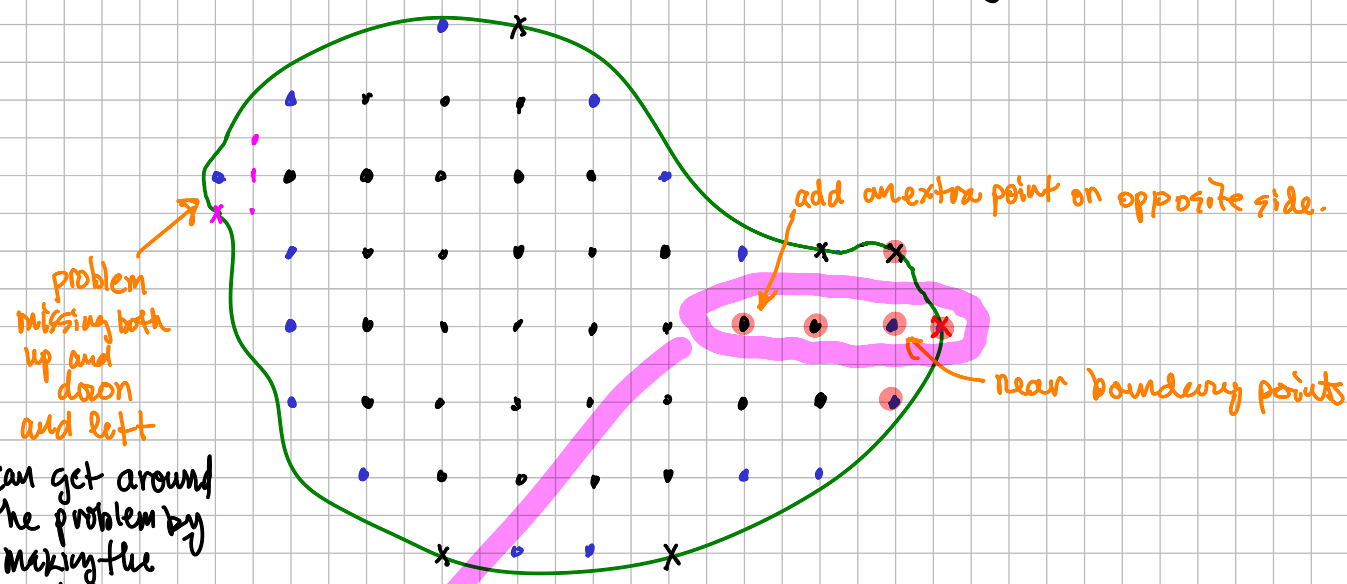
Look at the 8th disc

$$S_8 = \{ z \in \mathbb{C} : |-4 - z| \leq 1+1+1 \}$$

Note that $0 \notin S_8$ and in particular $0 \notin \partial S_8$. So 0 is not eigenvalue of A. So A is invertible.

Note this argument is general enough that it can be used to show A is invertible even when the domain includes near boundary points... in the following way:

What to do when there are near boundary points



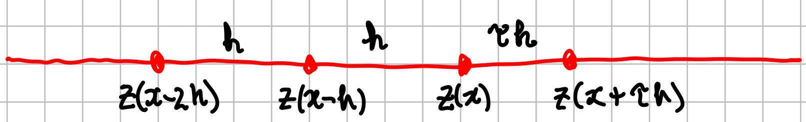
... can get around the problem by making the grid finer.

... and even finer so two points fit on other side.

Consider the case where the stencil doesn't fit in the domain, but at least one point in each direction does fit... actually need 2 points in the direction that fits...

Consider just one dimension

where $\tau \in (0,1)$



want to approximate $z'(x)$. How?

Taylor series
to solve for
a, b, c and d in $z''(x) \approx a\,z(x-2h) + b\,z(x-h) + c\,z(x) + d\,z(x+\tau h)$

$$a\,z(x-2h) = a\left(z(x) - 2h\,z'(x) + \frac{4h^2}{2} z''(x) - \frac{8h^3}{3!} z'''(x) + O(h^4)\right)$$

$$b\,z(x-h) = b\left(z(x) - h\,z'(x) + \frac{h^2}{2} z''(x) - \frac{h^3}{3!} z'''(x) + O(h^4)\right)$$

$$c\,z(x) = c\,z(x)$$

$$d\,z(x+\tau h) = d\left(z(x) + \tau h\,z'(x) + \frac{\tau^2 h^2}{2} z''(x) + \frac{\tau^3 h^3}{3!} z'''(x) + O(h^4)\right)$$

Choose a, b, c, d so these terms cancel
and so this one sums to 1.

$$a + b + c + d = 0$$
$$-2a - b + \tau d = 0$$
$$4a + b + \tau^2 d = 1$$
$$-8 - b + \tau^3 d \approx 0$$

Solve 4 linear equations in 4 unknowns to find the approximation at a near boundary point.