Solve the ODE ... $y \in \mathbb{R}^n$

$$\underbrace{y' = f(t,y)}_{\text{general term}} \quad t \geq t_0 \quad y(t_0) = y_0$$

general term — need hypothesis on f.

① f Lipschitz cont. in the second variable ...

$$|f(t,x) - f(t,y)| \leq \lambda |x-y|$$

called Lipschitz constant $\lambda > 0$

note any differentiable function f satisfies this..

Alternatively.

② f is analytic if y ... infinite # of derivatives in y and the power series (Taylor series) converges...

Theorem

- Under either hypothesis the ODE has a unique solution at least on a short interval of time $[t_0, t_0+\varepsilon]$.
- If the solution remains bounded then the interval of existence can be extended to $[t_0, \infty)$.

Two possibilities:

~~① There is no solution.~~ Theorem tells there is a solution

② I don't know what the solution is.

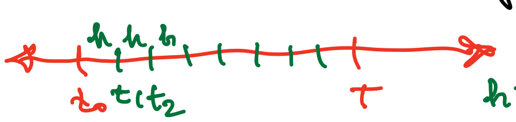
Seek to approximate the solution numerically...

ODE IVP

initial value problem...

$$y' = f(t,y), \quad t \geq t_0 \quad y(t_0) = y_0$$

Though the solution may exist on an infinite interval $[t_0, \infty)$ we consider approximations on the finite interval $[t_0, T]$



$h = \frac{T - t_0}{N}$ where N is the # of subintervals

To approximate solution introduce a grid of times

$$t_n = t_0 + hn \quad \text{where } h = \frac{T - t_0}{N}.$$

For the approximation of y we write

$$y(t_n) \approx y_n$$

actual solution that we don't know

numerical approximation we are calculating ...

Error in the approximation.

$$e_n = y_n - y(t_n)$$

note $e_n \in \mathbb{R}^n$

approximation

exact value

Size of the error is $\|e_n\|$

$$y' = f(t,y) \qquad y(t_0) = y_0 \qquad t \geq t_0$$

Integrate over the grid subintervals

$$\int_{t_0}^{t_1} y'\,dt = \int_{t_0}^{t_1} f(t,y)\,dt$$

fund. theorem of calculus

now approximate the integral

$h = t_1 - t_0$

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t,y)\,dt \approx f(t_0, y(t_0))\,(t_1 - t_0)$$

height of rectangle

width of rectangle

Thus

$$\boxed{y(t_1) \approx y(t_0) + h f(t_0, y(t_0))}$$

This motivates setting

$$y_1 = y_0 + h f(t_0, y_0)$$

Iteration gives

$$y_2 = y_1 + h f(t_1, y_1)$$

In general that

$$\boxed{y_{n+1} = y_n + h f(t_n, y_n)}$$ Called Euler's method

Questions:

① Does the approximation converge to the exact solution as one takes the limit $N \to \infty$ so the grid gets finer..

② How fast is the convergence? what are bounds on the error in the approximation.

What's the error between $y(t)$ and y_n on $[t_0, T]$?

we already defined $e_n = y_n - y(t_n)$

$$\max \{ \|e_n\| : n = 1, \dots, N \}$$ & size of the error in the approximation over the whole interval $[t_0, T]$

The method converges on $[t_0, T]$ if

$$\lim_{N\to\infty} \left(\max \{ \|e_n\| : n = 1, \dots, N \} \right) = 0$$

How to prove Euler's method is convergent...

Consider now e_{n+1} is related to e_n...

$$e_{n+1} = y_{n+1} - y(t_{n+1}) = y_n + h f(t_n, y_n) - y(t_{n+1})$$

approximation of $y(t_n)$

$$= y_n - y(t_n) + h f(t_n, y_n) - y(t_{n+1}) + y(t_n)$$

$$= e_n + h f(t_n, y_n) \underbrace{- y(t_{n+1}) + y(t_n)}$$

need to compare these terms.

Use Taylors theorem to express $y(t_{n+1})$ expanded about $t = t_n$.

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + O(h^2)$$

Since $y' = f(t,y)$ then we can substitute

$$y(t_{n+1}) = y(t_n) + h f(t_n, y(t_n)) + O(h^2)$$

now compare these two ...

$$e_{n+1} = e_n + h f(t_n, y_n) - \left(\cancel{y(t_n)} + h f(t_n, y(t_n)) \right)^{+O(h^2)} + \cancel{y(t_n)}$$

Thus,

$$e_{n+1} = e_n + h f(t_n, y_n) - h f(t_n, y(t_n)) + O(h^2)$$

compare these using Lipschitz cond. on f.

by hypothesis $|f(t,x) - f(t,y)| \le \lambda |x - y|$

$$\|e_{n+1}\| \le \|e_n\| + h\lambda \underbrace{\|y_n - y(t_n)\|}_{\text{thats } e_n \text{ again}} = (1 + h\lambda)\|e_n\| + O(h^2)$$

Now have a recursive estimate of the errors

$$\|e_{n+1}\| \le (1 + h\lambda)\|e_n\| + O(h^2)$$

important term this is where the error is coming from.

Now use induction to estimate $\|e_n\|$ ...

next time...