Exercise 3.1 Find the order of the following quadrature formulae:

a $\int_0^1 f(\tau)d\tau = \frac{1}{6}f(0) + \frac{2}{3}f(\frac{1}{2}) + \frac{1}{6}f(1)$ (the Simpson rule):

Solution: By definition the order of the formula is p where the quadrature matches the integral exactly whenever f is an arbitrary polynomial of degree $p-1$. Therefore we consider the monomials $f(t) = t^n$ and look for the smallest positive n such that the quadrature formula does not match.

Case $n = 0$. Then $f(t) = 1$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 1d\tau = 1$$

along with the fact that

$$\tfrac{1}{6}f(0) + \tfrac{2}{3}f(\tfrac{1}{2}) + \tfrac{1}{6}f(1) = \tfrac{1}{6} + \tfrac{2}{3} + \tfrac{1}{6} = \tfrac{6}{6} = 1$$

implies that the order is at least 1.

Case $n = 1$. Then $f(t) = t$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 td\tau = \tfrac{1}{2}$$

along with the fact that

$$\tfrac{1}{6}f(0) + \tfrac{2}{3}f(\tfrac{1}{2}) + \tfrac{1}{6}f(1) = \tfrac{1}{6}\cdot 0 + \tfrac{2}{3}\cdot\tfrac{1}{2} + \tfrac{1}{6}\cdot 1 = \tfrac{3}{6} = \tfrac{1}{2}$$

implies that the order is at least 2.

Case $n = 2$. Then $f(t) = t^2$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^2d\tau = \tfrac{1}{3}$$

along with the fact that

$$\tfrac{1}{6}f(0) + \tfrac{2}{3}f(\tfrac{1}{2}) + \tfrac{1}{6}f(1) = \tfrac{1}{6}\cdot 0^2 + \tfrac{2}{3}\cdot\tfrac{1}{2^2} + \tfrac{1}{6}\cdot 1^2 = \tfrac{4}{12} = \tfrac{1}{3}$$

implies that the order is at least 3.

Case $n = 3$. Then $f(t) = t^3$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^3 d\tau = \tfrac{1}{4}$$

along with the fact that

$$\tfrac{1}{6}f(0) + \tfrac{2}{3}f(\tfrac{1}{2}) + \tfrac{1}{6}f(1) = \tfrac{1}{6} \cdot 0^3 + \tfrac{2}{3} \cdot \tfrac{1}{2^3} + \tfrac{1}{6} \cdot 1^3 = \tfrac{3}{12} = \tfrac{1}{4}$$

implies that the order is at least 4.

Case $n = 4$. Then $f(t) = t^4$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^4 d\tau = \tfrac{1}{5}$$

along with the fact that

$$\tfrac{1}{6}f(0) + \tfrac{2}{3}f(\tfrac{1}{2}) + \tfrac{1}{6}f(1) = \tfrac{1}{6} \cdot 0^4 + \tfrac{2}{3} \cdot \tfrac{1}{2^4} + \tfrac{1}{6} \cdot 1^4 = \tfrac{5}{24} \neq \tfrac{1}{5}$$

implies that the order is exactly 4.

c $\displaystyle\int_0^1 f(\tau)d\tau = \tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4})$:

Solution: Check $f(t) = t^n$ for increasing values of n as before.

Case $n = 0$. Then $f(t) = 1$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 1 d\tau = 1$$

along with the fact that

$$\tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4}) = \tfrac{2}{3} - \tfrac{1}{3} + \tfrac{2}{3} = \tfrac{3}{3} = 1$$

implies that the order is at least 1.

Case $n = 1$. Then $f(t) = t$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t d\tau = \tfrac{1}{2}$$

along with the fact that

$$\tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4}) = \tfrac{2}{3}\cdot\tfrac{1}{4} - \tfrac{1}{3}\cdot\tfrac{1}{2} + \tfrac{2}{3}\cdot\tfrac{3}{4} = \tfrac{6}{6} = 1$$

implies that the order is at least 2.

Case $n = 2$. Then $f(t) = t^2$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^2 d\tau = \tfrac{1}{3}$$

along with the fact that

$$\tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4}) = \tfrac{2}{3}\cdot\tfrac{1}{4^2} - \tfrac{1}{3}\cdot\tfrac{1}{2^2} + \tfrac{2}{3}\cdot\tfrac{3^2}{4^2} = \tfrac{16}{48} = \tfrac{1}{3}$$

implies that the order is at least 3.

Case $n = 3$. Then $f(t) = t^3$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^3 d\tau = \tfrac{1}{4}$$

along with the fact that

$$\tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4}) = \tfrac{2}{3}\cdot\tfrac{1}{4^3} - \tfrac{1}{3}\cdot\tfrac{1}{2^3} + \tfrac{2}{3}\cdot\tfrac{3^3}{4^3} = \tfrac{48}{192} = \tfrac{1}{4}$$

implies that the order is at least 4.

Case $n = 4$. Then $f(t) = t^4$ and

$$\int_0^1 f(\tau)d\tau = \int_0^1 t^4 d\tau = \tfrac{1}{5}$$

along with the fact that

$$\tfrac{2}{3}f(\tfrac{1}{4}) - \tfrac{1}{3}f(\tfrac{1}{2}) + \tfrac{2}{3}f(\tfrac{3}{4}) = \tfrac{2}{3}\cdot\tfrac{1}{4^4} - \tfrac{1}{3}\cdot\tfrac{1}{2^4} + \tfrac{2}{3}\cdot\tfrac{3^4}{4^4} = \tfrac{37}{192} \neq \tfrac{1}{5}$$

implies that the order is exactly 4.

d $\displaystyle\int_0^\infty f(\tau)e^{-\tau}d\tau = \tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4).$

Solution: Check $f(t) = t^n$ for increasing values of n as before.

Case $n = 0$. Then $f(t) = 1$ and

$$\int_0^\infty f(\tau)e^{-\tau}d\tau = \int_0^\infty e^{-\tau}d\tau = -e^{-\tau}\Big|_0^\infty = 1$$

along with the fact that

$$\tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4) = \tfrac{5}{3} - \tfrac{3}{2} + 1 - \tfrac{1}{6} = \tfrac{6}{6} = 1$$

implies that the order is at least 1.

Case $n = 1$. Then $f(t) = t$ and

$$\begin{aligned}\int_0^\infty f(\tau)e^{-\tau}d\tau &= \int_0^\infty \tau e^{-\tau}d\tau = -\int_0^\infty \tau de^{-\tau}\\ &= -\tau e^{-\tau}\Big|_0^\infty + \int_0^\infty e^{-\tau}d\tau = 0 + 1 = 1\end{aligned}$$

along with the fact that

$$\tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4) = \tfrac{5}{3}\cdot 1 - \tfrac{3}{2}\cdot 2 + 1\cdot 3 - \tfrac{1}{6}\cdot 4 = \tfrac{3}{3} = 1$$

implies that the order is at least 2.

Case $n = 2$. Then $f(t) = t^2$ and

$$\begin{aligned}\int_0^\infty f(\tau)e^{-\tau}d\tau &= \int_0^\infty \tau^2 e^{-\tau}d\tau = -\int_0^\infty \tau^2 de^{-\tau}\\ &= -\tau^2 e^{-\tau}\Big|_0^\infty + \int_0^\infty 2\tau e^{-\tau}d\tau = 0 + 2 = 2\end{aligned}$$

along with the fact that

$$\tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4) = \tfrac{5}{3}\cdot 1^2 - \tfrac{3}{2}\cdot 2^2 + 1\cdot 3^2 - \tfrac{1}{6}\cdot 4^2 = 2$$

implies that the order is at least 3.

Case $n = 3$. Then $f(t) = t^3$ and

$$\begin{aligned}\int_0^\infty f(\tau)e^{-\tau}d\tau = \int_0^\infty \tau^3 e^{-\tau}d\tau = -\int_0^\infty \tau^3 de^{-\tau} \\ = -\tau^3 e^{-\tau}\Big|_0^\infty + \int_0^\infty 3\tau^2 e^{-\tau}d\tau = 0 + 6 = 6\end{aligned}$$

along with the fact that

$$\tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4) = \tfrac{5}{3}\cdot 1^3 - \tfrac{3}{2}\cdot 2^3 + 1\cdot 3^3 - \tfrac{1}{6}\cdot 4^3 = 6$$

implies that the order is at least 4.

Case $n = 4$. Then $f(t) = t^4$ and

$$\begin{aligned}\int_0^\infty f(\tau)e^{-\tau}d\tau = \int_0^\infty \tau^4 e^{-\tau}d\tau = -\int_0^\infty \tau^4 de^{-\tau} \\ = -\tau^4 e^{-\tau}\Big|_0^\infty + \int_0^\infty 4\tau^3 e^{-\tau}d\tau = 0 + 24 = 24\end{aligned}$$

along with the fact that

$$\tfrac{5}{3}f(1) - \tfrac{3}{2}f(2) + f(3) - \tfrac{1}{6}f(4) = \tfrac{5}{3}\cdot 1^4 - \tfrac{3}{2}\cdot 2^4 + 1\cdot 3^4 - \tfrac{1}{6}\cdot 4^4 = 16 \neq 24$$

implies that the order is exactly 4.

Exercise 3.4 Restricting your attention to scalar autonomous equations $y' = f(y)$, prove that the ERK method with tableau

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

is of order 4.

Solution: First write this scheme algebraically as

$$
\begin{aligned}
k_1 &= f(y_n) \\
k_2 &= f\big(y_n + \tfrac{1}{2}hk_1\big) \\
k_3 &= f\big(y_n + \tfrac{1}{2}hk_2\big) \\
k_4 &= f\big(y_n + hk_3\big) \\
y_{n+1} &= y_n + h\big(\tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4\big)
\end{aligned}
$$

Define $\beta_j(t_n)$ to be the result when the exact solution $y(t_n)$ is substituted into k_j. Thus

$$
\begin{aligned}
\beta_1(t) &= f\big(y(t)\big) \\
\beta_2(t) &= f\big(y(t) + \tfrac{1}{2}h\beta_1(t)\big) \\
\beta_3(t) &= f\big(y(t) + \tfrac{1}{2}h\beta_2(t)\big) \\
\beta_4(t) &= f\big(y(t) + h\beta_3(t)\big)
\end{aligned}
$$

It follows that the truncation error is

$$
\psi_n = y(t_{n+1}) - y(t_n) - h\big(\tfrac{1}{6}\beta_1(t_n) + \tfrac{1}{3}\beta_2(t_n) + \tfrac{1}{3}\beta_3(t_n) + \tfrac{1}{6}\beta_4(t_n)\big)
$$

To show the method is order 4 we need to show $\psi_n = \mathcal{O}(h^5)$. We now expand all terms up to the required order about $t = t_n$.

For simplicity, following the notation in the book, we write f, f_y, f_{yy} and f_{yyy} with no arguments to denote $f(y(t_n))$, $f_y(y(t_n))$, $f_{yy}(y(t_n))$ and $f_{yyy}(y(t_n))$ respectively. Similarly write y and β_i for $y(t_n)$ and $\beta_i(t_n)$.

First note that

$$\begin{aligned} y'(t) &= f \\ y''(t) &= f_y f \\ y'''(t) &= f_{yy} f^2 + f_y^2 f \\ y^{(4)}(t) &= f_{yyy} f^3 + 2 f_{yy} f_y f^2 + 2 f_{yy} f_y f^2 + f_y^3 f \\ &= f_{yyy} f^3 + 4 f_{yy} f_y f^2 + f_y^3 f. \end{aligned}$$

Consequently

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + h y'(t_n) + \tfrac{h^2}{2} y''(t_n) + \tfrac{h^3}{3!} y'''(t_n) + \tfrac{h^4}{4!} y^{(4)}(t_n) + \mathcal{O}(h^5) \\ &= y + hf + \tfrac{h^2}{2} f_y f + \tfrac{h^3}{3!} f_{yy} f^2 + \tfrac{h^3}{3!} f_y^2 f \\ &\quad + \tfrac{h^4}{4!} f_{yyy} f^3 + \tfrac{h^4}{3!} f_{yy} f_y f^2 + \tfrac{h^4}{4!} f_y^3 f + \mathcal{O}(h^5). \end{aligned}$$

Furthermore,

$$\beta_1 = f$$

and

$$\begin{aligned} \beta_2 &= f\big(y + \tfrac{1}{2} h \beta_1\big) \\ &= f + \tfrac{1}{2} h \beta_1 f_y + \tfrac{1}{2^3} h^2 \beta_1^2 f_{yy} + \tfrac{1}{2^3 \cdot 3!} h^3 \beta_1^3 f_{yyy} + \mathcal{O}(h^4). \end{aligned}$$

and similarly

$$\begin{aligned} \beta_3 &= f\big(y + \tfrac{1}{2} h \beta_2\big) \\ &= f + \tfrac{1}{2} h \beta_2 f_y + \tfrac{1}{2^3} h^2 \beta_2^2 f_{yy} + \tfrac{1}{2^3 \cdot 3!} h^3 \beta_2^3 f_{yyy} + \mathcal{O}(h^4). \end{aligned}$$

Substituting β_1 into β_2 and β_2 into β_3 yields

$$\beta_2 = f + \tfrac{1}{2} h f_y f + \tfrac{1}{2^3} h^2 f_{yy} f^2 + \tfrac{1}{2^3 \cdot 3!} h^3 f_{yyy} f^3 + \mathcal{O}(h^4).$$

and

$$\begin{aligned} \beta_3 &= f + \tfrac{1}{2} h \big\{ f + \tfrac{1}{2} h f_y f + \tfrac{1}{2^3} h^2 f_{yy} f^2 \big\} f_y \\ &\quad + \tfrac{1}{2^3} h^2 \big\{ f + \tfrac{1}{2} h f_y f \big\}^2 f_{yy} + \tfrac{1}{2^3 \cdot 3!} h^3 f_{yyy} f^3 + \mathcal{O}(h^4). \end{aligned}$$

Collecting terms then yields

$$\begin{aligned} \beta_3 &= f + \tfrac{1}{2} h f_y f + \tfrac{1}{2^2} h^2 \big\{ f_y^2 f + \tfrac{1}{2} f_{yy} f^2 \big\} \\ &\quad + \tfrac{1}{2^3} h^3 \big\{ \tfrac{1}{2} f_{yy} f_y f^2 + f_{yy} f_y f^2 + \tfrac{1}{3!} f_{yyy} f^3 \big\} + \mathcal{O}(h^4) \\ &= f + \tfrac{1}{2} h f_y f + \tfrac{1}{2^2} h^2 \big\{ f_y^2 f + \tfrac{1}{2} f_{yy} f^2 \big\} \\ &\quad + \tfrac{1}{2^3} h^3 \big\{ \tfrac{3}{2} f_{yy} f_y f^2 + \tfrac{1}{3!} f_{yyy} f^3 \big\} + \mathcal{O}(h^4) \end{aligned}$$

Finally

$$\begin{aligned}\beta_4 &= f\big(y + h\beta_3\big)\\ &= f + h\beta_3 f_y + \tfrac{1}{2}h^2\beta_3^2 f_{yy} + \tfrac{1}{3!}h^3\beta_3^3 f_{yyy} + \mathcal{O}(h^4)\\ &= f + h\big\{f + \tfrac{1}{2}hf_yf + \tfrac{1}{2^2}h^2\big(f_y^2 f + \tfrac{1}{2}f_{yy}f^2\big)\big\}f_y\\ &\quad + \tfrac{1}{2}h^2\big\{f + \tfrac{1}{2}hf_yf\big\}^2 f_{yy} + \tfrac{1}{3!}h^3 f^3 f_{yyy} + \mathcal{O}(h^4)\end{aligned}$$

Again collecting terms yields

$$\begin{aligned}\beta_4 = f &+ hf_yf + \tfrac{1}{2}h^2\big\{f_y^2 f + f_{yy}f^2\big\}\\ &+ \tfrac{1}{2^2}h^3\big\{f_y^3 f + \tfrac{1}{2}f_{yy}f_yf^2 + 2f_{yy}f_yf^2 + \tfrac{2}{3}f^3 f_{yyy}\big\} + \mathcal{O}(h^4)\end{aligned}$$

It follows that

$$\begin{aligned}\psi_n = y &+ hf + \tfrac{h^2}{2}f_yf + \tfrac{h^3}{3!}f_{yy}f^2 + \tfrac{h^3}{3!}f_y^2 f\\ &+ \tfrac{h^4}{4!}f_{yyy}f^3 + \tfrac{h^4}{3!}f_{yy}f_yf^2 + \tfrac{h^4}{4!}f_y^3 f - y\\ &- h\tfrac{1}{6}f - h\tfrac{1}{3}\big\{f + \tfrac{1}{2}hf_yf + \tfrac{1}{2^3}h^2 f_{yy}f^2 + \tfrac{1}{2^3\cdot 3!}h^3 f_{yyy}f^3\big\}\\ &- h\tfrac{1}{3}\big\{f + \tfrac{1}{2}hf_yf + \tfrac{1}{2^2}h^2\big\{f_y^2 f + \tfrac{1}{2}f_{yy}f^2\big\}\\ &\qquad + \tfrac{1}{2^3}h^3\big(\tfrac{3}{2}f_{yy}f_yf^2 + \tfrac{1}{3!}f_{yyy}f^3\big)\big\}\\ &- h\tfrac{1}{6}\big\{f + hf_yf + \tfrac{1}{2}h^2\big\{f_y^2 f + f_{yy}f^2\big\}\\ &\qquad + \tfrac{1}{2^2}h^3\big(f_y^3 f + \tfrac{5}{2}f_{yy}f_yf^2 + \tfrac{2}{3}f^3 f_{yyy}\big)\big\}\\ &+ \mathcal{O}(h^5)\end{aligned}$$

We now verify that all terms cancel.

$$\begin{gathered} y\{1-1\} = 0\\ hf\big\{1 - \tfrac{1}{6} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{6}\big\} = 0\\ h^2 f_yf\big\{\tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{6} - \tfrac{1}{6}\big\} = 0\\ h^3 f_{yy}f^2\big\{\tfrac{1}{6} - \tfrac{1}{24} - \tfrac{1}{24} - \tfrac{1}{12}\big\} = 0\\ h^3 f_y^2 f\big\{\tfrac{1}{6} - \tfrac{1}{12} - \tfrac{1}{12}\big\} = 0\\ h^4 f_{yyy}f^3\big\{\tfrac{1}{24} - \tfrac{1}{144} - \tfrac{1}{144} - \tfrac{1}{36}\big\} = 0\\ h^4 f_{yy}f_yf^2\big\{\tfrac{1}{6} - \tfrac{1}{16} - \tfrac{5}{48}\big\} = 0\\ h^4 f_y^3 f\big\{\tfrac{1}{24} - \tfrac{1}{24}\big\} = 0\end{gathered}$$

Therefore $\psi_4 = \mathcal{O}(h^5)$ and the method has been shown to be order 4.

Exercise 3.7 Write the theta method

$$y_{n+1} = y_n + h\big[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\big]$$

as a Runge–Kutta method.

Solution: Consider the tableau

$$\begin{array}{c|cc} 0 & & \\ 1 & \theta & 1-\theta \\ \hline & \theta & 1-\theta \end{array}$$

and write the corresponding for the approximation z_n where $z_n \approx y(t_n)$ as

$$\begin{aligned} \xi_1 &= z_n \\ \xi_2 &= z_n + h\big[\theta f(t_n, \xi_1) + (1-\theta) f(t_n + h, \xi_2)\big] \\ z_{n+1} &= z_n + h\big[\theta f(t_n, \xi_1) + (1-\theta) f(t_n + h, \xi_2)\big]. \end{aligned}$$

To see this is the same as the theta method note that

$$z_{n+1} = z_n + h\big[\theta f(t_n, \xi_1) + (1-\theta) f(t_n + h, \xi_2)\big] = \xi_2.$$

Consequently $t_n + h = t_{n+1}$ implies

$$z_{n+1} = z_n + h\big[\theta f(t_n, z_n) + (1-\theta) f(t_{n+1}, z_{n+1})\big]$$

which is identical to the theta method.

Exercise 3.8 Derive the three-stage Runge–Kutta method that corresponds to the collocation points $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$ and $c_2 = \frac{3}{4}$. Then determine its order.

Solution: Use Lemma 3.5 from the text to find the tableau from

$$a_{ji} = \int_0^{c_j} \ell_i(\tau)d\tau \qquad \text{for} \qquad j, i = 1, 2, \ldots, \nu$$

and

$$b_j = \int_0^1 \ell_i(\tau)d\tau \qquad \text{for} \qquad j = 1, 2, \ldots, \nu$$

where $\ell_i(t)$ are the Lagrange basis functions given by

$$\ell_i(t) = \prod_{j \neq i} \frac{t - c_j}{c_i - c_j}.$$

Compute

$$\ell_1(t) = \frac{(t - \frac{1}{2})(t - \frac{3}{4})}{(\frac{1}{4} - \frac{1}{2})(\frac{1}{4} - \frac{3}{4})} = \frac{t^2 - \frac{5}{4}t + \frac{3}{8}}{\frac{1}{8}} = 8t^2 - 10t + 3,$$

$$\ell_2(t) = \frac{(t - \frac{1}{4})(t - \frac{3}{4})}{(\frac{1}{2} - \frac{1}{4})(\frac{1}{2} - \frac{3}{4})} = \frac{t^2 - t + \frac{3}{16}}{-\frac{1}{16}} = -16t^2 + 16t - 3,$$

$$\ell_3(t) = \frac{(t - \frac{1}{4})(t - \frac{1}{2})}{(\frac{3}{4} - \frac{1}{4})(\frac{3}{4} - \frac{1}{2})} = \frac{t^2 - \frac{3}{4}t + \frac{1}{8}}{\frac{1}{8}} = 8t^2 - 6t + 1.$$

Consequently,

$$b_1 = \int_0^1 \ell_1(\tau)d\tau = \int_0^1 (8t^2 - 10t + 3)d\tau = \frac{8}{3} - 5 + 3 = \frac{2}{3},$$

$$b_2 = \int_0^1 \ell_2(\tau)d\tau = \int_0^1 (-16t^2 + 16t - 3)d\tau = \frac{-16}{3} + 8 - 3 = -\frac{1}{3},$$

$$b_3 = \int_0^1 \ell_3(\tau)d\tau = \int_0^1 (8t^2 - 6t + 1)d\tau = \frac{8}{3} - 3 + 1 = \frac{2}{3}.$$

To save time use Julia to compute the rest of the integrals. The program

```
using Symbolics,SymbolicNumericIntegration

@variables t
ell=[8*t^2-10*t+3,-16*t^2+16*t-3,8*t^2-6*t+1]
c=[1//4,1//2,3//4]
for j=1:3
    for i=1:3
        P=integrate(ell[i])[1]
        aij=substitute(P,[t=>c[j]])-substitute(P,[t=>0])
        println("A[$j,$i]=",aij)
    end
end
```

produces the output

```
julia> include("finda.jl")
A[1,1]=23//48
A[1,2]=-1//3
A[1,3]=5//48
A[2,1]=7//12
A[2,2]=-1//6
A[2,3]=1//12
A[3,1]=9//16
A[3,2]=0//1
A[3,3]=3//16
```

which implies

$$a_{11} = \int_0^{1/4} (8t^2 - 10t + 3)d\tau = \frac{23}{48}$$

$$a_{12} = \int_0^{1/4} (-16t^2 + 16t - 3)d\tau = -\frac{1}{3}$$

$$a_{13} = \int_0^{1/4} (8t^2 - 6t + 1)d\tau = \frac{5}{48}$$

$$a_{21} = \int_0^{1/2} (8t^2 - 10t + 3)d\tau = \frac{7}{12}$$

$$a_{22} = \int_0^{1/2} (-16t^2 + 16t - 3)d\tau = -\frac{1}{6}$$

$$a_{23} = \int_0^{1/2} (8t^2 - 6t + 1)d\tau = \frac{1}{12}$$

$$a_{31} = \int_0^{3/4} (8t^2 - 10t + 3)d\tau = \frac{9}{16}$$

$$a_{32} = \int_0^{3/4} (-16t^2 + 16t - 3)d\tau = 0$$

$$a_{33} = \int_0^{3/4} (8t^2 - 6t + 1)d\tau = \frac{3}{16}.$$

The resulting tableau is

$$\begin{array}{c|ccc} \frac{1}{4} & \frac{23}{48} & -\frac{1}{3} & \frac{5}{48} \\ \frac{1}{2} & \frac{7}{12} & -\frac{1}{6} & \frac{1}{12} \\ \frac{3}{4} & \frac{9}{16} & 0 & \frac{3}{16} \\ \hline & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}.$$