Exercise 8.1 Prove the identities

$$\Delta_- + \Delta_+ = 2\Upsilon_0\Delta_0 \qquad \text{and} \qquad \Delta_-\Delta_+ = \Delta_0^2.$$

Solution: Since

$$\Delta_- z_k = z_k - z_{k-1} \qquad \text{and} \qquad \Delta_+ z_k = z_{k+1} - z_k$$

then

$$(\Delta_- + \Delta_+)z_k = z_k - z_{k-1} + z_{k+1} - z_k = z_{k+1} - z_{k-1}.$$

On the other hand

$$\begin{aligned} 2\Upsilon_0\Delta_0 z_k &= 2\Upsilon_0(z_{k+1/2} - z_{k-1/2}) = 2(\Upsilon_0 z_{k+1/2} - \Upsilon_0 z_{k-1/2}) \\ &= 2\Big(\frac{z_{k+1} + z_k}{2} - \frac{z_k + z_{k-1}}{2}\Big) = z_{k+1} - z_{k-1}. \end{aligned}$$

As both expressions evaluate to $z_{k+1} - z_{k-1}$ they must be equal.

To demonstrate the other identity is similar. First note

$$\begin{aligned} \Delta_-\Delta_+ z_k &= \Delta_-\big(z_{k+1} - z_k\big) \\ &= (z_{k+1} - z_k) - (z_k - z_{k-1}) = z_{k+1} - 2z_k + z_{k-1} \end{aligned}$$

while

$$\begin{aligned} \Delta_0^2 z_k &= \Delta_0(z_{k+1/2} - z_{k-1/2}) \\ &= (z_{k+1} - z_k) - (z_k - z_{k-1}) = z_{k+1} - 2z_k + z_{k-1}. \end{aligned}$$

Again both expressions evaluate to the same thing so they must be equal.

Exercise 8.3 Demonstrate that for every $s \geq 1$ there exists a constant $c_s \neq 0$ such that

$$\frac{d^s z(x)}{dx^s} - \frac{1}{h^s}\left(\Delta_+^s - \tfrac{1}{2}s\Delta_+^{s+1}\right)z(x) = c_s \frac{d^{s+2}z(x)}{dx^{s+2}}h^2 + \mathcal{O}(h^3) \quad \text{as} \quad h \to 0$$

for every sufficiently smooth function z. Evaluate c_s explicitly for $s = 1, 2$.

Solution: Since

$$\mathcal{D} = \frac{1}{h}\log(\mathcal{E}) = \frac{1}{h}\log(\mathcal{I} + \Delta_+) = \frac{1}{h}\left(\Delta_+ - \tfrac{1}{2}\Delta_+^2 + \tfrac{1}{3}\Delta_+^3\right) + \mathcal{O}(h^3)$$

it follows that

$$\begin{aligned} \mathcal{D}^s &= \frac{1}{h^s}\left(\Delta_+ - \tfrac{1}{2}\Delta_+^2 + \tfrac{1}{3}\Delta_+^3\right)^s + \mathcal{O}(h^3) \\ &= \frac{1}{h^s}\left(\Delta_+^s - \tfrac{1}{2}s\Delta_+^{s+1} + \tfrac{1}{24}s(3s+5)\Delta_+^{s+2}\right) + \mathcal{O}(h^3). \end{aligned}$$

Replacing s by $s+2$ yields

$$\begin{aligned} \mathcal{D}^{s+2} &= \frac{1}{h^{s+2}}\left(\Delta_+^{s+2} - \tfrac{1}{2}(s+2)\Delta_+^{s+3} + \tfrac{1}{24}(s+2)(3s+11)\Delta_+^{s+4}\right) + \mathcal{O}(h^3) \\ &= \frac{1}{h^{s+2}}\Delta_+^{s+2} + \mathcal{O}(h). \end{aligned}$$

Therefore

$$\frac{1}{h^s}\Delta_+^{s+2} = h^2\mathcal{D}^{s+2} + \mathcal{O}(h^3).$$

It follows that

$$\begin{aligned} \mathcal{D}^s - \frac{1}{h^s}\left(\Delta_+^s - \tfrac{1}{2}s\Delta_+^{s+1}\right) &= \frac{1}{h^s}s(3s+5)\Delta_+^{s+2} + \mathcal{O}(h^3) \\ &= s(3s+5)h^2\mathcal{D}^{s+2} + \mathcal{O}(h^3) \end{aligned}$$

and consequently $c_s = s(3s+5)$.

Explicitly we obtain $c_1 = 8$ and $c_2 = 22$.

Exercise 8.4 For every $s \geq 1$ find a constant $d_s \neq 0$ such that

$$\frac{d^{2s}z(x)}{dx^{2s}} - \frac{1}{h^{2s}}\Delta_0^{2s}z(x) = d_s\frac{d^{2s+2}z(x)}{dx^{2s+2}}h^2 + \mathcal{O}(h^4) \quad \text{as} \quad h \to 0$$

for every sufficiently smooth function z. Compare the sizes of d_1 and c_2. What does this tell you about the errors in the forward difference and central difference approximations?

Solution: Recall that

$$\mathcal{D} = \frac{2}{h}\log\left\{\tfrac{1}{2}\Delta_0 + \sqrt{\mathcal{I} + \tfrac{1}{4}\Delta_0^2}\right\}.$$

The same Taylor expansion used for equation (8.8) in the text taken to lower order then implies

$$\mathcal{D}^{2s} = \frac{1}{h^{2s}}\left\{\Delta_0^{2s} - \frac{s}{12}\Delta_0^{2s+2}\right\} + \mathcal{O}(h^4).$$

Replacing s by $s+2$ yields

$$\begin{aligned}\mathcal{D}^{2s+2} &= \frac{1}{h^{2s+2}}\left\{\Delta_0^{2s+2} - \frac{s+2}{12}\Delta_0^{2s+4}\right\} + \mathcal{O}(h^4)\\ &= \frac{1}{h^{2s+2}}\Delta_0^{2s+2} + \mathcal{O}(h^2).\end{aligned}$$

Therefore

$$\frac{1}{h^{2s}}\Delta_0^{2s+2} = h^2\mathcal{D}^{2s+2} + \mathcal{O}(h^4).$$

It follows that

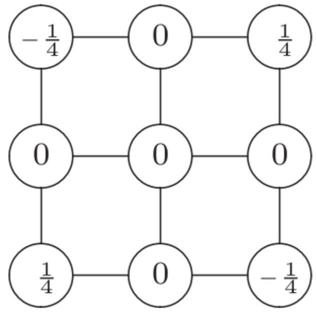
$$\mathcal{D}^{2s} - \frac{1}{h^{2s}}\Delta_0^{2s} = -\frac{1}{h^{2s}}\frac{s}{12}\Delta_0^{2s+2} + \mathcal{O}(h^4) = -\frac{s}{12}h^2\mathcal{D}^{2s+s} + \mathcal{O}(h^4)$$

and consequently $d_s = -s/12$. Explicitly $d_1 = -1/12$.

Since the forward difference formula for $s = 2$ approximates the second derivative, then comparing the size of c_2 to d_1 provides the coefficient on the first non-vanishing term in the Taylor series approximation, respectively, of the second derivative using forward and central differences. In particular $|c_2| = 22$ and $|d_1| = 1/12$ suggest the central difference formula will be about $|c_2/d_1| = 264$ times more accurate.

Exercise 8.6 Determine the order in the form $\mathcal{O}((\Delta x)^p)$ of the finite difference approximation to $\partial^2/\partial x \partial y$ given by the computational stencil

$$\frac{1}{(\Delta x)^2}$$



Solution: By definition the stencil approximation is

$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{1}{4(\Delta x)^2}\big(- u(x-\Delta x, y+\Delta x) + u(x+\Delta x, y+\Delta x) + u(x-\Delta x, y-\Delta x) - u(x+\Delta x, y-\Delta x)\big)$$

Now expand each term using Taylor series as

$$\begin{aligned} u(x-\Delta x, y+\Delta x) &= u(x,y) + \Delta x\Big(- \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y}\Big) + \mathcal{O}((\Delta x)^5) \\ &+ \frac{(\Delta x)^2}{2!}\Big(\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) \\ &+ \frac{(\Delta x)^3}{3!}\Big(- \frac{\partial^3 u}{\partial x^3} + \frac{3\partial^3 u}{\partial x^2 \partial y} - \frac{3\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3}\Big) \\ &+ \frac{(\Delta x)^4}{4!}\Big(\frac{\partial^4 u}{\partial x^4} - \frac{4\partial^4 u}{\partial x^3 \partial y} + \frac{6\partial^4 u}{\partial x^2 \partial y^2} - \frac{4\partial^4 u}{\partial x \partial y^3} + \frac{\partial^4 u}{\partial y^4}\Big), \end{aligned}$$

$$\begin{aligned} u(x+\Delta x, y+\Delta x) &= u(x,y) + \Delta x\Big(\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y}\Big) + \mathcal{O}((\Delta x)^5) \\ &+ \frac{(\Delta x)^2}{2!}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{2\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) \\ &+ \frac{(\Delta x)^3}{3!}\Big(\frac{\partial^3 u}{\partial x^3} + \frac{3\partial^3 u}{\partial x^2 \partial y} + \frac{3\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3}\Big) \\ &+ \frac{(\Delta x)^4}{4!}\Big(\frac{\partial^4 u}{\partial x^4} + \frac{4\partial^4 u}{\partial x^3 \partial y} + \frac{6\partial^4 u}{\partial x^2 \partial y^2} + \frac{4\partial^4 u}{\partial x \partial y^3} + \frac{\partial^4 u}{\partial y^4}\Big), \end{aligned}$$

$$
\begin{aligned}
u(x-\Delta x, y-\Delta x) = u(x,y) &+ \Delta x\Big(-\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\Big) + \mathcal{O}((\Delta x)^5) \\
&+ \frac{(\Delta x)^2}{2!}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{2\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) \\
&+ \frac{(\Delta x)^3}{3!}\Big(-\frac{\partial^3 u}{\partial x^3} - \frac{3\partial^3 u}{\partial x^2 \partial y} - \frac{3\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}\Big) \\
&+ \frac{(\Delta x)^4}{4!}\Big(\frac{\partial^4 u}{\partial x^4} + \frac{4\partial^4 u}{\partial x^3 \partial y} + \frac{6\partial^4 u}{\partial x^2 \partial y^2} + \frac{4\partial^4 u}{\partial x \partial y^3} + \frac{\partial^4 u}{\partial y^4}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
u(x+\Delta x, y-\Delta x) = u(x,y) &+ \Delta x\Big(\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\Big) + \mathcal{O}((\Delta x)^5) \\
&+ \frac{(\Delta x)^2}{2!}\Big(\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\Big) \\
&+ \frac{(\Delta x)^3}{3!}\Big(\frac{\partial^3 u}{\partial x^3} - \frac{3\partial^3 u}{\partial x^2 \partial y} + \frac{3\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}\Big) \\
&+ \frac{(\Delta x)^4}{4!}\Big(\frac{\partial^4 u}{\partial x^4} - \frac{4\partial^4 u}{\partial x^3 \partial y} + \frac{6\partial^4 u}{\partial x^2 \partial y^2} - \frac{4\partial^4 u}{\partial x \partial y^3} + \frac{\partial^4 u}{\partial y^4}\Big).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
-u(x-\Delta x, y+\Delta x) + u(x+\Delta x, y+\Delta x) = &\ \Delta x\Big(2\frac{\partial u}{\partial x}\Big) \\
&+ \frac{9\Delta x)^2}{2!}\Big(\frac{4\partial^2 u}{\partial x \partial y}\Big) + \frac{(\Delta x)^3}{3!}\Big(\frac{2\partial^3 u}{\partial x^3} + \frac{6\partial^3 u}{\partial x \partial y^2}\Big) \\
&+ \frac{(\Delta x)^4}{4!}\Big(\frac{8\partial^4 u}{\partial x^3 \partial y} + \frac{8\partial^4 u}{\partial x \partial y^3}\Big) + \mathcal{O}((\Delta x)^5)
\end{aligned}
$$

and

$$
\begin{aligned}
u(x-\Delta x, y-\Delta x) - u(x+\Delta x, y-\Delta x) = &\ \Delta x\Big(-\frac{2\partial u}{\partial x}\Big) \\
&+ \frac{(\Delta x)^2}{2!}\Big(\frac{4\partial^2 u}{\partial x \partial y}\Big) + \frac{(\Delta x)^3}{3!}\Big(-\frac{2\partial^3 u}{\partial x^3} - \frac{6\partial^3 u}{\partial x \partial y^2}\Big) \\
&+ \frac{(\Delta x)^4}{4!}\Big(\frac{8\partial^4 u}{\partial x^3 \partial y} + \frac{8\partial^4 u}{\partial x \partial y^3}\Big) + \mathcal{O}((\Delta x)^5).
\end{aligned}
$$

Adding the above equations together yields

$$
\begin{aligned}
&-u(x-\Delta x, y+\Delta x)+u(x+\Delta x, y+\Delta x) \\
&\quad u(x-\Delta x, y-\Delta x)-u(x+\Delta x, y-\Delta x) \\
&\qquad\qquad = 4(\Delta x)^2 \frac{\partial^2 u}{\partial x \partial y} + \tfrac{2}{3}(\Delta x)^4 \Big(\frac{\partial^4 u}{\partial x^3 \partial y} + \frac{\partial^4 u}{\partial x \partial y^3} \Big) + \mathcal{O}((\Delta x)^5).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{1}{4(\Delta x)^2} \big(&-u(x-\Delta x, y+\Delta x)+u(x+\Delta x, y+\Delta x) \\
&u(x-\Delta x, y-\Delta x)-u(x+\Delta x, y-\Delta x)\big) \\
&= \frac{\partial^2 u}{\partial x \partial y} + \tfrac{1}{6}(\Delta x)^2 \Big(\frac{\partial^4 u}{\partial x^3 \partial y} + \frac{\partial^4 u}{\partial x \partial y^3} \Big) + \mathcal{O}((\Delta x)^3),
\end{aligned}
$$

which implies $p = 2$.