Heat equation:

What is a PDE? An equation with partial derivatives in it...

Example:

$$\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$$

Mathematical approach: Look at the equation and try to figure out what the solutions do.

Physical approach: Study a physical problem (i.e. look at the solutions) and then write down some differential equations describing what's going on.

Historically: The physical approach is where many interesting PDE's came from...

Heat conduction in a 1-dimensional rod.

$e(x,t)$ thermal energy density...

Energy is: Calories, Joules, BTUs

per unit volume: $/cm^3$, $/ft^3$, $/m^3$

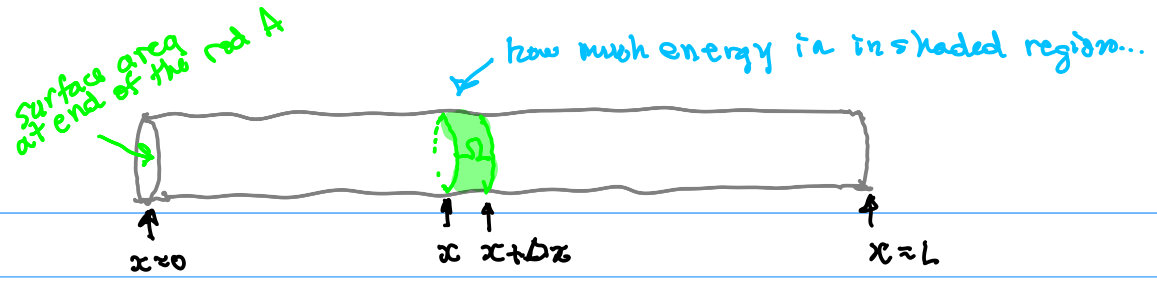
amount of energy to raise one gram of water by one degree celsius. → Calories

amount of energy to raise a pound of water from freezing to boiling... → BTUs

Watch the units:

$[E]$	units of energy.	Joules, BTUs
$[L]^3 = [V]$	units of volume	m^3
$[L]$	units of length	m
$[T]$	units of time	sec
$[L]^2 = [A]$	units of area	m^2
$[M]$	units of mass	g
$[u]$	units of temerature	$^\circ C$.

rod.



$e(x,t)$ thermal energy density...

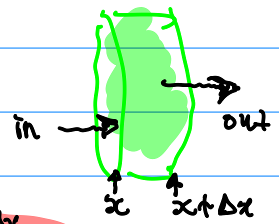
Energy $= \int_{\Omega} e\, dV = A\int_{x}^{x+\Delta x} e(s,t)\,ds$

rate of change of heat energy in time	=	heat energy flowing across boundaries per unit time	+	heat energy generated inside per unit time.

Heat flowing across boundaries:

Heat flux: ϕ units $\frac{[E]}{[T][L]^2}$

Heat source: Q units $\frac{[E]}{[T][L]^3}$



rate of energy flowing from left to right...

$$\frac{d}{dt} A\int_{x}^{x+\Delta x} e(s,t)\,ds = A\phi(x,t) - A\phi(x+\Delta x,t) + \int_{x}^{x+\Delta x} Q(s,t)\, A\,ds$$

Turn this into a differential equation... by taking $\Delta x \to 0$

First order approximations:

$$\frac{\partial}{\partial t} e(x,t)\cancel{A}\underbrace{\Delta x}_{0} = \cancel{A}\phi(x,t) - \cancel{A}\underbrace{\phi(x+\Delta x,t)}_{0} + Q(x,t)\cancel{A}\underbrace{\Delta x}_{0}$$

Note if I take $\Delta x \to 0$ in the above equation ... I get

$$0 = A\phi(x,t) - A\phi(x,t) + 0 = 0$$

Since $0=0$ is not useful divide by Δx first then take limit.

$$\frac{\partial}{\partial t} e(x,t) = \underbrace{\frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}}_{\text{tends to a derivative in } x} + Q(x,t)$$

Now take limit: $\Delta x \to 0$

$$\boxed{\frac{\partial}{\partial t} e(x,t) = -\frac{\partial}{\partial x}\phi(x,t) + Q(x,t)}$$

Maybe I know what Q is... that's the heating element under the rod... but I don't know $e(x,t)$ or $\phi(x,t)$...

Two unknowns — one equation.

- Need a relationship between e and ϕ to make this a solvable equation..

Fourier's Law:

Before I do that... lets convert energy into temperature...

$e(x,t)$ $\qquad \frac{[E]}{[L]^3}$

Changes in temperatures are used to define the energy...

$\boxed{\text{amount of energy to raise one gram of water by one degree celsius.}}$ → Energy is Calories

Joules

amount of energy to raise a pound of water from freezing to boiling... → BTUs

Heat capacity to convert temperatures to energies...

$c(x)$ heat capacity of the rod at position x.

units of c $\qquad \frac{[E]}{[M][u]}$ $\qquad$ the amount of energy per unit mass per degree...

Just knowing the units allows me to figure out the relationship between temperature and energy in the bar..

$$\frac{[E]}{[L]^3} = \frac{[E]}{[M][\mu]} \; [\mu] \; \frac{[M]}{[L]^3}$$

energy density — heat capacity — temperature — density

Let $\rho(x)$ be the density of the rod at x.

$$\boxed{e(x,t) = c(x)\,u(x,t)\,\rho(x)}$$

relationship between temperature and energy..

Substitute this into the differential equation

$$\frac{\partial}{\partial t} e(x,t) = -\frac{\partial}{\partial x} \phi(x,t) + Q(x,t)$$

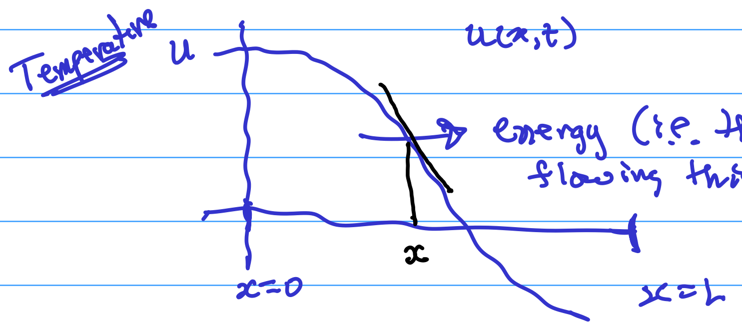
$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + Q(x,t)$$

So far I converted energies into temperatures. Next need to convert the flux ϕ in to temperatures...

Back to...

Fourier's Law:

hot things cool off and cold things warm up in proportion to the temperature differences...



energy (i.e. the heat) is flowing this way..., in proportion to the change of temperature.

$\frac{\partial u}{\partial x}$

Fourier's law:

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

$$\frac{[E]}{[T][L]^2} = \frac{[E]}{[T][U][L]} \quad \frac{[U]}{[L]}$$

The dimensions of K_0 is $\dfrac{[E]}{[T][U][L]}$

energy per degree difference per length per unit time.

Called the conductivity..

Next substitute for ϕ in terms of u to get the heat equation...