The Heat Equation:

e — energy density units: $\frac{[E]}{[L]^3}$

ϕ ~ energy flux units: $\frac{[E]}{[T][L]^2}$

Q — rate of energy production units: $\frac{[E]}{[T][L]^3}$

After taking $\Delta x \to 0$ we obtained

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

Convert e and ϕ into temperatures...

Definition of heat energy — Heat capacity.

$$e(x,t) = c(x)\rho(x)\,u(x,t)$$

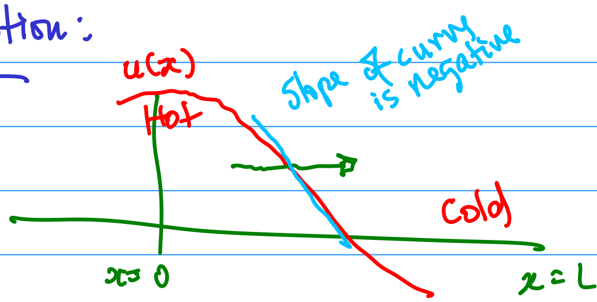
heat capacity, density, temperature

Fourier's law idea that hot flows towards cold...

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

conductivity, temperature difference.

Sign convention:



flux moves from left to right which means opposite the direction of $\frac{\partial u}{\partial x}$.

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

$$e(x,t) = c(x)\rho(x)u(x,t)$$

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$ ← inside the bar...

Substituting yield...

Heat equation:

$$\boxed{c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = +\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u(x,t)}{\partial x}\right) + Q}$$

Want to use the heat equation to predict the future. To do this need more information:

<u>Physical point of view:</u>

1. Know the current state of the bar... Initial condition... initial temperature distribution.
2. Energy flux passing in and out of the bar at the left and the right - Boundary conditions...
3. Also need to know rate of energy production Q. This is a forcing.
4. Also need to know conductivity, density and heat capacity...

Simplify the problem ... for illustration....

$$c\cancel{(x)}\rho\cancel{(x)}\frac{\partial u(x,t)}{\partial t} = +\frac{\partial}{\partial x}\left(K_0\cancel{(x)}\frac{\partial u(x,t)}{\partial x}\right) \cancel{+Q}$$

Assume the conductivity, density and the heat capacity are constant (i.e. don't depend on x) throughout the bar.

Also that no heat is being producted so $Q=0$.

$$\boxed{c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}}$$ ← simpler heat equation

$$\frac{\partial u}{\partial t} = \frac{K_0}{c\rho} \frac{\partial^2 u}{\partial x^2}$$

Thus, $k = \frac{K_0}{c\rho}$ diffusivity (rate of diffusion of heat)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Mathematically what's needed to predict the future...

1. need to integrate the time... one integral since first order in time, One integration means one constant of integration. That constant is the initial temperature distribution.

2. still need the flux ϕ at $x=0$ and $x=L$.

3. Need to know the constant k.

What are the dimensional units of k the diffusivity.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

what are the units of $\frac{\partial u}{\partial t}$ units: $\frac{[u]}{[T]}$

$\frac{\partial^2 u}{\partial x^2}$ units $\frac{[u]}{[L]^2}$

Thus k has dimensions $\frac{[L]^2}{[T]}$.

Note also $k = \frac{K_0}{c\rho}$ and I already figured out the dimensial measurements for K_0, c and ρ.

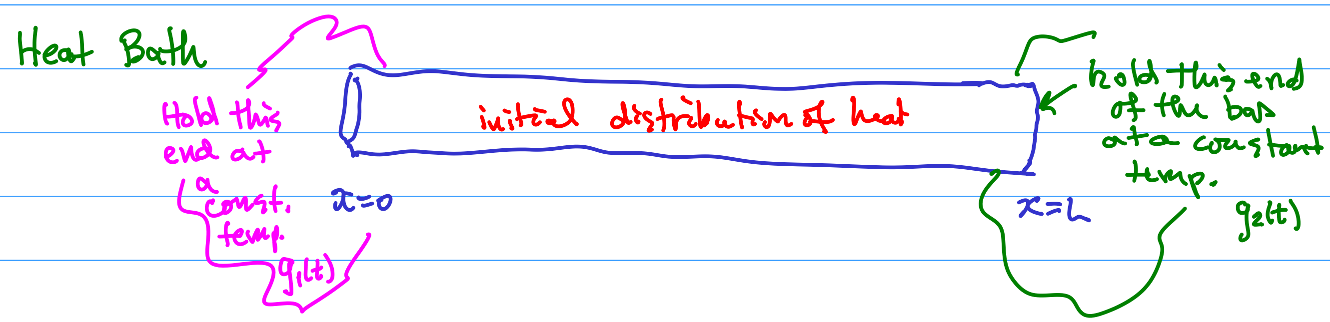
- Idea check that all the dimensions are consistent at home...

To predict the future we need

- The PDE $\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$
- The initial conditions
- The boundary conditions

The initial condition is obvious (just measure the temperature of the bar at each point x). There is nothing else to do...

Boundary condition come in different types:



Boundary conditions: $u(0,t)=g_1(t)$ and $u(L,t)=g_2(t)$

Insulated Boundary condition... Vacuum at each end so no heat can enter or leave the bar.

Flux is zero at each end of the bar.

$$\phi(0,t)=0 \qquad\qquad \phi(L,t)=0$$

flux is $\frac{[E]}{[T][L]^2}$

Thus $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ and $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

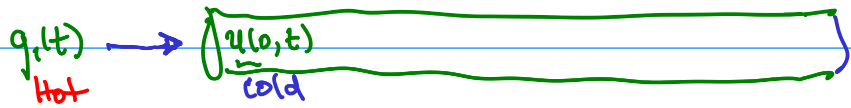
Newton's law of cooling...

H_1 is a constant

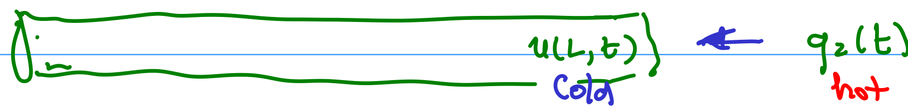
boundary conditions
$$\begin{cases} \phi(0,t) = H_1\left(g_1(t) - u(0,t)\right) \\ \phi(L,t) = H_2\left(u(L,t) - g_2(t)\right) \end{cases}$$

const...

If $g_1(t) > u(0,t)$ then the flux flows left to right i.e. $\phi > 0$



If $g_2(t) > u(L,t)$ need $\phi < 0$ in this case



$$\phi(L,t) = H_2\left(\underbrace{u(L,t) - g_2(t)}_{\text{negative}}\right)$$

negative so $\phi(L,t) < 0$

$$-K_0 \left.\frac{\partial u}{\partial x}\right|_{x=0} = \phi(0,t) = H_1\left(g_1(t) - u(0,t)\right)$$

(note page 12 section 1.3.4)

$$-K_0 \left.\frac{\partial u}{\partial x}\right|_{x=L} = \phi(L,t) = H_2\left(u(L,t) - g_2(t)\right)$$

PREDICT super far into the future as $t \to \infty$

- look for equilibrium solutions to the heat equation...

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

initial distribution of heat...

Initial condition $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = g_1$ and $u(L,t) = g_2$

assume bndy doesn't change in time...

Note an equilibrium solution is one that results as $t \to \infty$ (and doesn't change in time).

Thinking about $t \to \infty$ but solving for $\frac{\partial u}{\partial t} = 0$

Thus... $0 = k \frac{\partial^2 u}{\partial x^2}$ where $u(t,x) = u(x)$

Get an ordinary differential equation

$$\frac{d^2 u}{dx^2} = 0 \qquad u(0) = g_1 \qquad u(L) = g_2$$

functions that satisfy this ODE are lines.

$u(x) = mx + b$.

$u(0) = g_1$ means $b = g_1$

$u(x) = mx + g_1$

$u(L) = mL + g_1 = g_2 \qquad m = \frac{g_2 - g_1}{L}$

Solution is $\boxed{u(x) = \frac{g_2 - g_1}{L} x + g_1}$

Thus we've predicted the final state at $t \to \infty$. But how does the energy and the heat move to get to this final distribution... next week and the week after...

What about $t \to \infty$ for the insulated case. Then what is the final distribution of heat?

Total heat in the bar $= \int_0^L u \, dx$