What happens as $t \to \infty$ with insulated boundary?

PDE: $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

IC: $$u(x,0) = f(x)$$ ← initial heat distribution at time $t=0$.

$0$ ↑ time

BC: $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$ ← insulated boundary conditions, ie. energy flux is zero at the ends of the rod.

Assume at $t \to \infty$ an equilibrium state is obtained

$$u(x) = \lim_{t \to \infty} u(x,t)$$

↑ the equilibrium state is stationary in time...

Thus $\frac{\partial u}{\partial t} = 0$

Consequently

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{reduces to} \quad 0 = k \frac{\partial^2 u}{\partial x^2}$$

This leads to the ordinary differential equation

$$\frac{d^2 u}{dx^2} = 0$$

Same ODE as with the heat bath...

Solution $\quad u(x) = mx + b.$

Use boundary conditions to solve for the constants...

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \qquad \left.\frac{du}{dx}\right|_{x=L} = 0$$

Left bndry condition

$$\left.\frac{dy}{dx}\right|_{x=0} = u'(x) = m = 0$$

Right boundary condition is dependent, so I can't use it to solve for the other constant...

Thus all I know is that $u(x) = b$ but I don't know what $b$ is.

Since we believe the physical problem has a unique solution, because of the repeatability of the outcome of experiements, then we want a unique value for $b$.

Find it some other way... ?

IC: $\quad u(x, \underset{\text{time}}{0}) = f(x)$

Haven't used the initial condition. Maybe that can tell me what the value for $b$ is.

Since the ends are insulated no energy flux enters or leaves the bar. Thus, the total energy is constant. Since the temperature is proportional to the energy with respect to a reference energy corresponding to temperature 0. Then

$$e = c\rho u$$

total temperature $\int_0^L A u(x,t)\,dx$ doesn't depend on time.

Thus

$$\int_0^L A u(x,0)\,dx = \lim_{t\to 0} \int_0^L A u(x,0)\,dx$$

initial distribution of heat

$u(x,0) = f(x)$

time

equilibrium distribution of heat

$u(x) = b$

$$\int_0^L f(x)\,dx = \int_0^L b\,dx = Lb$$

Solve for $b$ gives

$$b = \frac{1}{L}\int_0^L f(x)\,dx$$

This finishes 1.4

## Chapter 1.5 : Higher dimensional heat problems..

Fundamental theorem of Calculus:

$\partial[a,b] = \{a,b\}$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

integral over the line

Something evaluated over the boundary of the line

Fundamental theorem of vector calculus..

In 3D

$$\int_R \nabla \cdot g \, dV = \int_{\partial R} g \cdot \hat{n} \, dS$$

integral over a volume

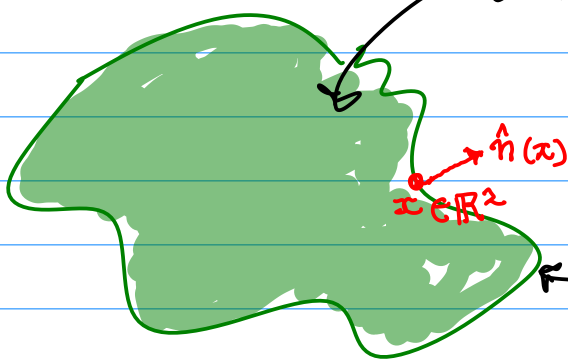
something evaluated over the boundary of the volume.

outward pointing unit normal vector to the surface

means the boundary of R.

Works the same in 2D and other dimensions as well...,

R is the filled in area....




boundaris is $\partial R$

## The Heat Equation:

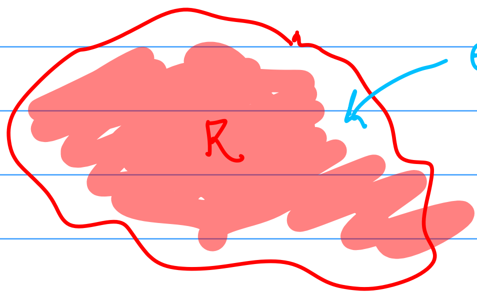
$e$ ~ energy density  units: $\frac{[E]}{[L]^3}$

$\phi$ ~ energy flux  units: $\frac{[E]}{[T][L]^2}$

$Q$ – rate of energy production  units: $\frac{[E]}{[T][L]^3}$

Conservation of energy...

Consider a Volume $R \subseteq \mathbb{R}^3$

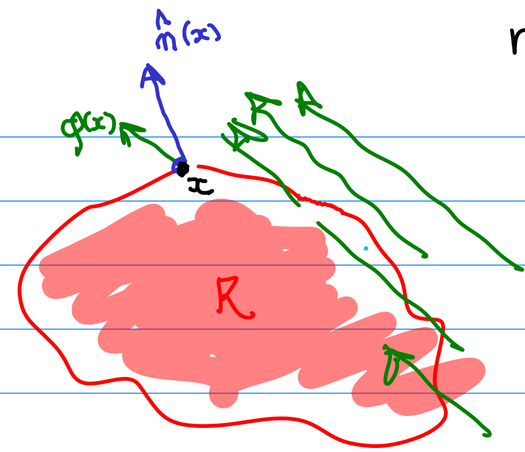



energy inside $\int_R e\,dV$

$$\frac{\partial}{\partial t}\int_R e\,dV = -\underbrace{\text{rate energy passes out through the boundary}}_{\text{total flux}} + \underbrace{\text{rate that energy is being produced inside}}_{\text{total production}}$$

Flux is the rate that the energy is moving about

Thus $\phi \in \mathbb{R}^3$




rate energy is leaving at the point $x$ is $\phi(x)\cdot\hat{n}(x)$

Total rate of energy leaving the volume R is

$$\int_{\partial R} \phi(x)\cdot\hat{n}(x)\,dS$$

Total production rate of energy inside is

$$\int_R Q\,dV$$

Plug all these terms into the conservation of energy...

$$\frac{\partial}{\partial t}\int_R e\,dV = -\underbrace{\text{rate energy passes out through the boundary}}_{\text{total flux}} + \underbrace{\text{rate that energy is being produced inside}}_{\text{total production}}$$

Thus

$$\frac{\partial}{\partial t}\int_R e\,dV = -\int_{\partial R}\phi\cdot\hat{n}\,dS + \int_R Q\,dV$$

The divergence theorem (fundamental theorem of Calculus)

$$\int_R \nabla\cdot\phi\,dV = \int_{\partial R}\phi\cdot\hat{n}\,dS$$

Conservation of energy...

$$\int_R \left( \frac{\partial}{\partial t} e + \nabla \cdot \varphi - Q \right) = 0$$

holds for every reasonable volume R
for which the divergence theorem holds..

If the integral is zero for every R, then

$$\frac{\partial e}{\partial t} + \nabla \cdot \varphi - Q = 0$$

only functions of $x$ (assumption)

As before: $\quad e = c\rho u \quad$ heat capacity...

and $\quad \varphi = -K_0(x) \nabla u \quad$ Fourier's law

Thus

$$\boxed{c\rho \frac{\partial u}{\partial t} - \nabla \cdot (K_0 \nabla u) = Q}$$

heat equation in 2 or 3 dimensions or more...

Assume $c(x) = c$, $\rho(x) = \rho$ and $K_0(x) = K_0$

$$c\rho \frac{\partial u}{\partial t} - K_0 \nabla^2 u = Q$$

Assume $Q = 0$

$$\frac{\partial u}{\partial t} = k \nabla^2 u \qquad k = \frac{K_0}{c\rho} \ldots$$

diffusivity...