Heat equation in n dimensions:

$$\boxed{c\rho \frac{\partial u}{\partial t} - \nabla \cdot (K_0 \nabla u) = Q}$$

Initial Conditions: Obvious $u(x,0) = u_0(x)$ for $x \in R$

Boundary conditions:

boundary of R

Heat bath: $u(x,t) = g(x)$ for $x \in \partial R$

Insulated: $\phi(x,t) \cdot \hat{n}(x) = 0$ for $x \in \partial R$

Fourier's Law.

$\phi(x,t) = -K_0(x) \nabla u(x,t)$

$\nabla u(x,t) \cdot \hat{n}(x) = 0$ for $x \in \partial R$.

Newton's Law of Cooling:

$\phi(x,t) \cdot \hat{n}(x) = H(u(x,t) - g(x,t))$ for $x \in \partial R$.

Predict the future: Equilibrium solution as $t \to \infty$...

then u doesn't change in time as it's in equilibrium. So $\frac{\partial u}{\partial t} = 0$

Heat equation

$$\cancel{c\rho\frac{\partial u}{\partial t}} - \nabla\cdot(K_0\nabla u) = Q$$

before crossing out $\frac{\partial u}{\partial t}$ this was a parabolic problem.

$$-\nabla\cdot(K_0\nabla u) = Q$$

Simplifying assumption: The conductivity is constant throughout the region R. $K_0(x) = K_0$

$$-K_0\nabla\cdot\nabla u = Q$$

time independent Q

Poisson equation...

$$\boxed{\nabla^2 u = -\frac{Q(x)}{K_0}}$$

unlike the 1-D case this is not an ODE but still a PDE

This is an elliptic problem ... i.e. statics problem equilibrium state

Set $Q=0$ then

Laplace equation

$$\boxed{\nabla^2 u = 0}$$

Laplace operator... in some books $\nabla^2 = \Delta$

Chapter 2 Seperation of Variables ...

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

simplest 1D heat equation

I.C. $u(x,0) = u_0(x)$ for $x \in [0,L]$

B.C. $u(x,t)=0$ for $x=0$ and $x=L$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

This is a Linear PDE...

In linear algebra we had matrices. What was a matrix? A linear function...

In PDE we have derivatives. What is a derivative? A linear operation on functions...

works like the matrix "u" $\rightarrow$ $L(u) = \frac{\partial u}{\partial t} - k\cdot\frac{\partial^2 u}{\partial x^2}$

heat distributions

$$L(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1+c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1+c_2u_2)$$

const const

$$= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - c_1k\frac{\partial^2}{\partial x^2}u_1 - c_2k\frac{\partial^2}{\partial x^2}u_2$$

$$= c_1L(u_1) + c_2L(u_2)$$

Thus $L(c_1u_1+c_2u_2) = c_1L(u_1) + c_2L(u_2)$

That means L is a linear operator on the space of heat distributions.

In linear algebra we solve

$Ax=b$

In PDEs we solve

$$L(u) = \frac{Q}{c\rho}$$

← both are linear problem.

Linear algebra: Consider the Homogeneous problem $Ax=0$
Solutions were called $\mathrm{Nul}(A) = \{x : Ax = 0\}$

PDEs: Consider the Homogeneous problem $Lu=0$
Solutions are the $\mathrm{Ker}(L) = \{u : Lu = 0\}$.

Seperation of variables

Assume $u(x,t) = \varphi(x)\,G(t)$

and plug it in...

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\Big(\varphi(x)\,G(t)\Big) = \varphi(x)\,\frac{d}{dt}G(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\Big(\varphi(x)\,G(t)\Big) = G(t)\,\frac{d^2}{dx^2}\varphi(x)$$

$$\varphi(x)\frac{d}{dt}G(t) = k\, G(t)\frac{d^2}{dx^2}\varphi(x)$$

Magic

$$\frac{1}{k\,G(t)}\frac{d}{dt}G(t) = \frac{1}{\varphi(x)}\frac{d^2}{dx^2}\varphi(x) = -\lambda$$

since the left doesn't depend on x
and the right doesn't depend on t
then since the are equal ...
they don't depend neither x or t

Therefore

$$\frac{1}{k\,G(t)}\frac{dG(t)}{dt} = -\lambda \qquad \frac{dG}{dt} = -\lambda k G$$

and

$$\frac{1}{\varphi(x)}\frac{d^2}{dx^2}\varphi(x) = -\lambda \qquad \frac{d^2}{dx^2}\varphi = -\lambda\varphi$$

these are like the eigenvalue-eigenvector equation $Ax = \lambda x$ in linear algebra...

These eigenvalue problems involving differential operators are ODE's and I already know how to solve them

Solve this ODE ...

$$\frac{dG}{dt} = -\lambda k G$$

Guess method $G(t) = G(0)\, e^{-\lambda k t}$

Seperation of vbls...,

$$\frac{dG}{dt} = -\lambda k G$$

$$\int \frac{dG}{G} = \int -\lambda k \, dt$$

$$\ln G = -\lambda k t + c$$

$$G = e^{\ln G} = e^{-\lambda k t + c} = e^{-\lambda k t} \underbrace{e^{c}}_{G(0)}$$

Thus

$$G(t) = G(0)\, e^{-\lambda k t}$$

Solve the other ODE:

$$\frac{d^2}{dx^2} \varphi = -\lambda \varphi$$

B.C. $u(x,t) = 0$ for $x = 0$ and $x = L$

$u(x,t) = \varphi(x)\, G(t) = 0$ for $x = 0$, $x = L$ and all time

Boundary value ODE

Then $\phi(0)=0$ and $\phi(L)=0$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{such that } \phi(0)=0 \text{ and } \phi(L)=0$$

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$\phi(0)=0 = A = 0$$

$$\phi(L)=0 = B\sin(\sqrt{\lambda}\,L) = 0$$

set of integers

thus $\sqrt{\lambda}\,L = n\pi$ for some $n \in \mathbb{Z}$

Solve for the eigenvalues....

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

To solve for B we will use initial condition

$$u(x,0) = u_0(x)$$

Next time...