Last time:

PDE $\quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

→ BC. $\quad u(0,t)=0 \quad u(L,t)=0 \quad$ for $t \geq 0$

IC. $\quad u(x,0) = f(x) \quad$ for $x \in [0,L]$.

Since the PDE is linear sums of solutions which which satisfy the PDE also satisfy the PDE.

Since the bounary conditions are homogeneous then sums of solutions which satisfy the boundary still satisfy the boundary condition.

Sums of solutions don't satisfy the initial condition. Actual a good thing... we'll choose our sum carefully so in the end it does satisfy the intial condition.

Separation of Variables

Assume $u(x,t) = \varphi(x)\, G(t)$

This separates the variables when plugged in to the PDE..

Led to the eigenfunction - eigenvalue problems

$$\frac{dG}{dt} = -\lambda k G$$

$$\frac{d^2}{dx^2}\varphi = -\lambda \varphi$$

Found the eigenfunctions and eigenvalues by solving the ODEs:.

$$G(t) = G(0)e^{-\lambda k t}$$

$$\varphi(x) = B\sin(\sqrt{\lambda}\,x)$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

Now use sums of solutions of the form $\varphi(x)G(t)$ in order to satisfy the Initial conditions.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Now this $u(x,t)$ satisfies the PDE because it's a sum of solution that satisfy the PDE. It also satisfies the boundary condition because the boundary conditions are homogeneous and each solution satisfied those boundary conditions.

What's left try to solve for the B_n's so the initial condition is also satisfied

Theory of Fourier Series,,,

Idea $u(x,0) = f(x)$

is this satisfied?

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

now solve for the B_n so the equation holds

Magic that makes this possible: Orthogonality..

Orthogonality relation between eigenfunctions

$$\int_0^L \boxed{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)} dx$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\frac{\partial}{\partial\alpha}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \boxed{\sin\alpha\sin\beta}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \boxed{\sin\alpha\sin\beta}$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

$$\frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} = \sin\alpha\sin\beta$$

$$\int_0^L \boxed{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)} dx = \frac{1}{2}\int_0^L\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx$$

$$= \begin{cases} \frac{L}{2} & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$ ← orthogonality relation.

Therefore

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = f(x)\sin\frac{m\pi x}{L}$$

$$\int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

$$\sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx}_{\text{orthogonality. the only term that survives is } n=m} \approx \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

solving for the B_m and this is now true.

$$B_m \frac{L}{2} = \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

Therefore

$$B_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

All the steps used to find B_m have to be reversible if I want to conclude that now this holds...

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \approx f(x)$$

Reflect solving for the B_n's was possible because of the orthogonality of the eigenfunctions...

Analogy with linear algebra...

$$\underbrace{Ax = \lambda x}_{\text{eigenvalue eigenvector problem}}$$

x — eigenvector
λ — eigenvalue.

If A is symmetric, what happens?

$A^T = A$

$Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ and $\lambda_1 \neq \lambda_2$

$$\lambda_1 x \cdot y = Ax \cdot y = (Ax)^T y = x^T A^T y$$

$$= x^T A y = x^T \lambda_2 y = \lambda_2 x \cdot y$$

Thus $\lambda_1 x \cdot y = \lambda_2 x \cdot y$ solve for $x \cdot y$

$$(\lambda_1 - \lambda_2) x \cdot y = 0$$

$$x \cdot y = \frac{0}{\lambda_1 - \lambda_2} = 0$$

$$x \cdot y = 0$$

Moreover the eigenvector of A form a basis.

Spectral theorem: If $A = A^T$ then A has a basis of orthonormal eigenvectors...

This is the same kind of orthogonality I want for the eigenfunctions

"This is what Sturm–Liouville theory is about.

Example:

PDE $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

BC. $u(0,t) = 0$ $u(L,t) = 0$ for $t \geq 0$

IC. $u(x,0) = 100$ for $x \in [0, L]$.

$f(x) = 100$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{2}{L}\int_0^L 100\sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Bigg|_0^L$$

@ $x = L$ $\quad \cos\frac{n\pi x}{L} = \cos\frac{n\pi L}{L} = \cos n\pi = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases}$

$$B_n = -\frac{200}{n\pi}\left(\begin{Bmatrix} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{Bmatrix} - 1\right) = \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Therefor the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\boxed{u(x,t) = \sum_{\substack{n \text{ odd} \\ n \geq 1}} \frac{400}{n\pi}\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

Solution.....

Note that as n gets bigger the exponential decay of the general term in the above sum is faster...

Note since the later terms decay faster than

$$u(x,t) \sim \frac{400}{\pi} \sin \frac{\pi x}{h} e^{-k\left(\frac{\pi}{h}\right)^2 t} \quad \text{as } t \to \infty$$