Last time we solved...

Chapter 2 Seperation of Variables ...

Simplest Heat Bath

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Simplest 1D heat equation

I.C. $u(x,0) = f(x)$ for $x \in [0, L]$

B.C. $u(x,t) = 0$ for $x = 0$ and $x = L$

Observation, since the PDE is linear and the B.C. are homogeneous then the superposition principle allows us to satisfy the initial condition using a sum of simple solutions of the form $\phi(x) G(t)$.

insulated boundary conditions

First example in section 2.4.

PDE $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

I.C. $u(x,0) = f(x)$ for $x \in [0, L]$

B.C. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

Try the same thing... see what the differences are and maybe eventually develop a general method..

Idea: separation of variables. Note superposition principle works since the PDE is still linear and the boundary conditions homogeneous...

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Plug in $u(x,t) = \phi(x)G(t)$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\phi(x)G(t)\right) = \phi(x)\frac{d}{dt}G(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\phi(x)G(t)\right) = G(t)\frac{d^2}{dx^2}\phi(x)$$

$$\frac{dG}{dt} = -\lambda k G \qquad\qquad \frac{d^2}{dx^2}\phi = -\lambda\phi$$

! Something about this differential operator that is like a symmetric matrix.

$$G(t) = G(0)e^{-\lambda k t}$$

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

BC. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ $\quad\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$ for PDE

BC. $\phi'(0) = 0$ $\quad\phi'(L) = 0$ for the ODE

$$\phi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\phi'(0) = B\sqrt{\lambda} = 0 \qquad B = 0$$

other boundary condition...

Linear Algebra: eigenvalue-eigenvector problem

n equations $\rightarrow$ $Ax = \lambda x$

Simultaneously solve for x and λ

$A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$

$1 + n$ unknowns

$$\phi'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

Thus $\sqrt{\lambda}L = n\pi$ where $n \in \mathbb{Z}$.

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

Therefore

$$\phi'(x) = -A\frac{n\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$$

$$\phi(x) = A\cos\left(\frac{n\pi}{L}x\right)$$

want eigen functions so can't be the zero function...

What happens when $n = 0$? $\phi(x) = A$

Now use sums of solutions of the form $\phi(x)G(t)$ in order to solve the initial condition

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Goal solve for the A_n's so $u(x,0) = f(x)$.

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

In PDE's our dot product was between functions rather than vectors...

$$\int_0^L x(t) \cdot y(t)\, dt$$

what plays the role of matrices? derivatives...

Why is transposing a matrix something useful to do with a linear function?

$$Ax \cdot y = (Ax)^T y$$

$$= x^T A^T y = x \cdot A^T y$$

Thus

$$Ax \cdot y = x \cdot A^T y$$

$$\frac{d^2}{dt^2} x \cdot y = \int_0^L x''(t)\, y(t)\, dt$$

solutions to the ODE

How to "hop" the differential operator over the dot in the dot product?

$u = y(t)$ $\quad du = y'(t)\,dt$ $\quad$ Integration by Parts

$dv = x''(t)\,dt$ $\quad v = x'(t)$

$$= \int_0^L u\,dv = uv\Big|_0^L - \int_0^L v\,du = \cancel{y(t)\,x'(t)}\Big|_0^L - \int_0^L x'(t)\,y'(t)\,dt$$

Boundary conditions $\quad \phi'(0) = 0 \quad \phi'(L) = 0$

$x'(0) = 0 \quad x'(L) = 0$

$y'(0) = 0 \quad y'(L) = 0$

$\Bigg\}$ assumption about $x(t)$ and $y(t)$.

$$= -\int_0^L x'(t)\,y'(t)\,dt =$$

$u = y'(t)$ $\quad du = y''(t)\,dt$

$dv = x'(t)\,dt$ $\quad v = x(t)$

$$= -\left[\cancel{x(t)\,y'(t)}\Big|_0^L - \int_0^L x(t)\,y''(t)\,dt\right] = \int_0^L x(t)\,y''(t)\,dt$$

$$= x \cdot \frac{d^2}{dt^2} y$$

Since $\frac{d^2}{dt^2}$ is self adjoint, i.e. it doesn't change when it hops over the dot product then the eigenfunctions are orthogonal...

The orthogonality allows solving for the An....

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

Take the dot product of the equation with $\cos \frac{m\pi x}{L}$.

$$\sum_{n=0}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\,dx = \int_0^L f(x) \cos \frac{m\pi x}{L}\,dx$$

Since cosine is the same as sine except for a phase shift, the same result holds as before... almost

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\,dx = \begin{cases} \frac{L}{2} & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0 \\ 0 & \text{if } n \neq m \end{cases}$$

But what about when $m = 0$ and $n = 0$...

difference than before is that $n=0$ is an eigenfunction...

Note, there are other dot products, for example

$$x(t) \cdot y(t) = \int_0^L x(t)\,y(t)\,w(t)\,dt$$

where $w(t) > 0$ is some weight function.

for later...

$$\sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx}_{\text{means } m=n} = \int_0^L f(x) \cos\frac{m\pi x}{L}\, dx$$

$$\frac{L}{2} A_m = \int_0^L f(x) \cos\frac{m\pi x}{L}\, dx \qquad \text{for } m \neq 0$$

$$L A_0 = \int_0^L f(x) \cos\frac{0\pi x}{L}\, dx$$

Therefore

$$A_m = \begin{cases} \frac{2}{L} \int_0^L f(x) \cos\frac{m\pi x}{L}\, dx & \text{for } m \neq 0 \\ \frac{1}{L} \int_0^L f(x)\, dx & \text{for } m = 0 \end{cases}$$

Next example

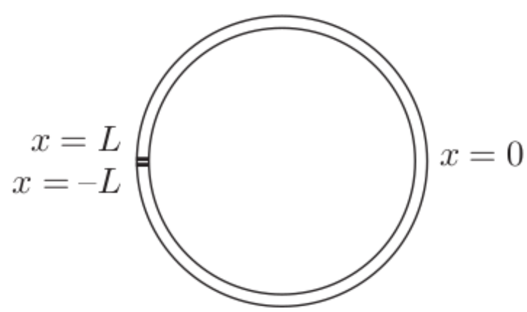

FIGURE 2.4.1 Thin circular ring.