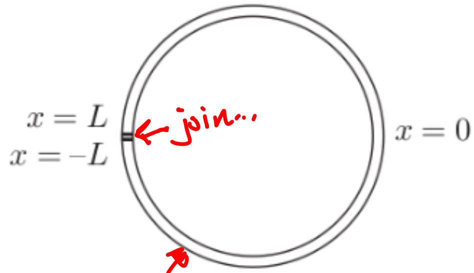


FIGURE 2.4.1 Thin circular ring.

parameterize the circle for $x \in [-L, L]$

Heat equation PDE: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ ← linear in u

initial condition is not homogeneous { IC: $u(x,0) = f(x)$ for $x \in [-L, L]$, $f(-L) = f(L)$

boundary is homogeneous { BC: $\underbrace{u(-L,t) = u(L,t)}$ linear in u — heat the same

$\phi(-L,t) = \phi(L,t)$ flux the same ← not the flux ϕ is different ~~than the~~

equivalently $\left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}$ ← linear in u, ϕ in the separation of vbls

So the superposition principle means I can (try) to add up a bunch of solutions while preserving the boundary conditions to achieve the initial condition...

Seperation of variables $u(x,t) = \phi(x) G(t)$ plug in and get

$$\frac{dG}{dt} = -\lambda k G \qquad\qquad \frac{d^2}{dx^2}\phi = -\lambda \phi$$

Solve

$G(t) = G(0) e^{-\lambda k t}$

as before

analyse this..

heat the same

$$\boxed{\varphi(-L) = \varphi(L)}$$

$$\frac{d^2}{dx^2}\varphi = -\lambda\varphi$$

flux is the same

$$\varphi'(-L) = \varphi'(L)$$

note the flux $\frac{\partial u}{\partial x}$ turned into φ'

General solution

$$\varphi(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$$

$$\varphi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\, x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\, x)$$

First

$$\varphi(-L) = \cancel{A\cos(\sqrt{\lambda}\, L)} - B\sin(\sqrt{\lambda}\, L)$$

$$\varphi(L) = \cancel{A\cos(\sqrt{\lambda}\, L)} + B\sin(\sqrt{\lambda}\, L)$$

$\varphi(-L) = \varphi(L)$ $\qquad$ $-B\sin(\sqrt{\lambda}\, L) = B\sin(\sqrt{\lambda}\, L)$

$$2B\sin(\sqrt{\lambda}\, L) = 0$$

either $B=0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$

Next

$$\varphi'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\, L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\, L)$$

$$\varphi'(-L) = A\sqrt{\lambda}\sin(\sqrt{\lambda}\, L) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\, L)$$

$\varphi'(-L) = \varphi'(L)$ either $B=0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$

$$-A\sqrt{\lambda}\sin(\sqrt{\lambda}\, L) = A\sqrt{\lambda}\sin(\sqrt{\lambda}\, L)$$

either $A=0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$

Together I have

(either $B=0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$) and (either $A=0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$)

need to be satisfied

Thus if $\sqrt{\lambda} \neq \frac{n\pi}{L}$ then both $A=0$ and $B=0$. But a zero function can't be an eigenfunction (just like the zero vector is not an eigenvector).

Therefor $\sqrt{\lambda} = \frac{n\pi}{L}$ and A and B are still to be solved for...

By the superposition principle

$$u(x,t) = \sum_{n=0}^{\infty} \phi_n(x)\, G_n(t)$$

$$= \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{n\pi}{L} x\right) + B_n \sin\left(\frac{n\pi}{L} x\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$= A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi}{L} x\right) + B_n \sin \frac{n\pi}{L} x \right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Solve for the constants using the initial condition $u(x,0) = f(x)$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi}{L} x\right) + B_n \sin \frac{n\pi}{L} x \right) = f(x)$$

Since $\frac{d^2}{dx^2}$ is "symmetric" actually self adjoint with respect to the "dot" product actually inner product

given by $\underset{x}{u(x) \cdot v(x)} = \int_{-L}^{L} u(x) v(x)\, dx$

usually written

$$(u, v) = \int_{-L}^{L} u(x) v(x)\, dx$$

Physics

$\langle u | v \rangle =$ — . . . , same thing ...

Then the eigenfunction are orthogonal. This allows solving for the coefficients easily...

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi}{L} x\right) + B_n \sin \frac{n\pi}{L} x \right) = f(x)$$

by multiplying and then integrating ... i.e. taking dot products...

note infinite dimensional

What we are doing is projecting f onto the basis of eigenfunctions of the differential operator $\frac{d^2}{dx^2}$...

mult by 1. and integrate, then what

$$\int_{-L}^{L}\left(A_0+\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{L}x\right)+B_n\sin\frac{n\pi}{L}x\right)\right)dx=\int_{-L}^{L}f(x)\,dx$$

$$\int_{-L}^{L}A_0+\sum_{n=1}^{\infty}\left(A_n\int_{-L}^{L}\cancel{\cos\left(\frac{n\pi}{L}x\right)}+B_n\int_{-L}^{L}\cancel{\sin\frac{n\pi}{L}x}\right)dx=\int_{-L}^{L}f(x)\,dx$$

$$2L\,A_0=\int_{-L}^{L}f(x)\,dx$$

$$\boxed{A_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx}$$

mult by $\cos\frac{m\pi}{L}x$ with $m\neq 0$ and integrate...

$$\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)\left(A_0+\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{L}x\right)+B_n\sin\frac{n\pi}{L}x\right)\right)dx=\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\,dx$$

by orthogonality

$$\int_{-L}^{L}A_m\left(\cos\frac{m\pi}{L}x\right)^2dx=\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\,dx$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$\frac{\partial}{\partial\alpha}$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\boxed{\sin\alpha\sin\beta}$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\boxed{\sin\alpha\sin\beta}$$

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

$$\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}=\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\cos\alpha\cos\beta$$

$$\frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} = \cos\alpha\cos\beta$$

$$(\cos\alpha)^2 = \frac{\cos(\alpha-\alpha) + \cos(\alpha+\alpha)}{2} = \frac{1+\cos 2\alpha}{2}$$

Therefore,

$$\int_{-L}^{L} A_m\left(\cos\frac{m\pi}{L}x\right)^2 dx = \int_{-L}^{L} A_m \frac{1+\cos\left(2\frac{m\pi}{L}x\right)}{2}\, dx = A_m L$$

and

$$\boxed{A_m = \frac{1}{L}\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\, dx}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\, dx$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\, dx.$$

similarly

$$\boxed{B_m = \frac{1}{L}\int_{-L}^{L}\left(\sin\frac{m\pi}{L}x\right)f(x)\, dx}$$

and from before

$$\boxed{A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx}$$

done with 2.4...

2.5

2.5 LAPLACE'S EQUATION: SOLUTIONS AND QUALITATIVE PROPERTIES

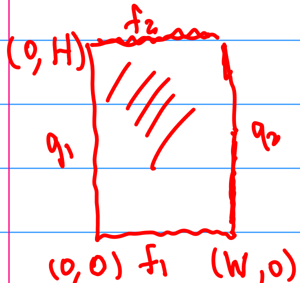
2.5.1 Laplace's Equation Inside a Rectangle

2D heat equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

$$\frac{\partial u}{\partial t} = k \nabla^2 u$$

Easiest boundary condition Heat bath on a rectangular domain.



$$u\left(\begin{bmatrix}0\\y\end{bmatrix}, t\right) = g_1(y) \qquad u\left(\begin{bmatrix}x\\0\end{bmatrix}, t\right) = f_1(x)$$

$$u\left(\begin{bmatrix}W\\y\end{bmatrix}, t\right) = g_2(y) \qquad u\left(\begin{bmatrix}x\\H\end{bmatrix}, t\right) = f_2(x)$$

Look for an equilibrium solution when $t \to \infty$. Then u does not depend on time $\frac{\partial u}{\partial t} = 0$

Laplace equation...

$$\nabla^2 u = 0$$

B.C.

$$u\left(\begin{bmatrix}0\\y\end{bmatrix}\right) = g_1(y) \qquad u\left(\begin{bmatrix}x\\0\end{bmatrix}\right) = f_1(x)$$

$$u\left(\begin{bmatrix}W\\y\end{bmatrix}\right) = g_2(y) \qquad u\left(\begin{bmatrix}x\\H\end{bmatrix}\right) = f_2(x)$$

Since Laplace equation is homogeneous, can add solution together and they still satisfy the PDE. Supperposition principle to satisfy the boundary condition...