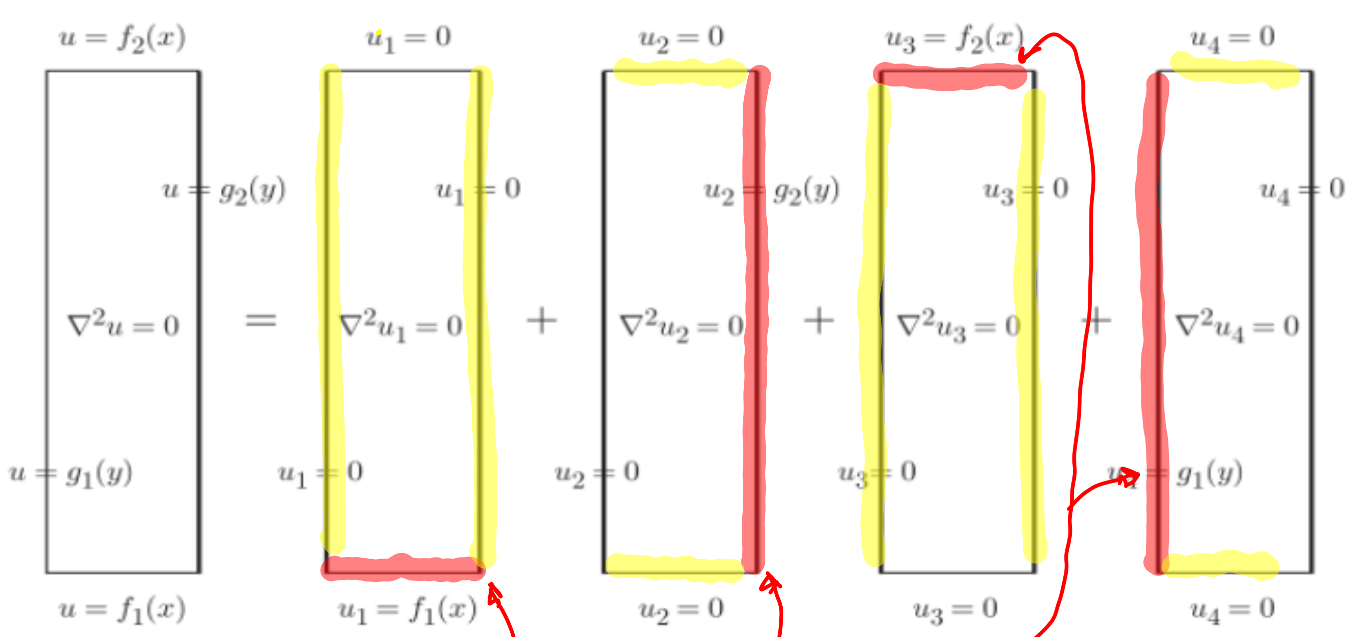
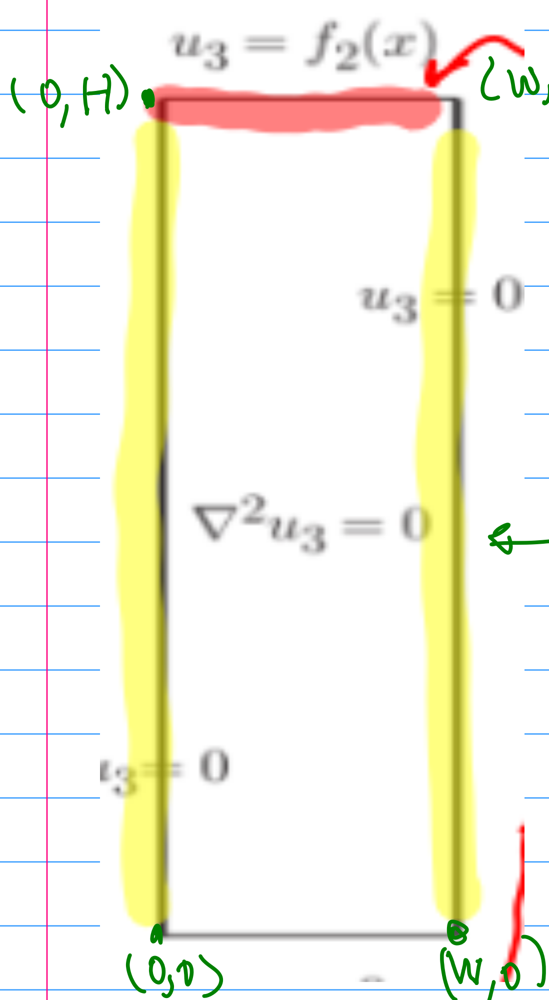


non-homogeneous boundaries



The book solves u_4 we'll solve u_3 for variety...

What is this problem?

PDE $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

this is the equilibrium solution of the heat equation in two dimensions

homogeneous BC

$u(0,y) = 0$

$u(W,y) = 0$

$u(x,0) = 0$

Inhomogeneous BC.

$u(x,H) = f_2(x)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

linear PDE thats also homogeneous...

corresponds to the homogeneous boundary condition

Separation of variables: $u(x,y) = \phi(x)\, w(y)$

$$\phi''(x)\, w(y) + \phi(x)\, w''(y) = 0$$

Thus
$$\frac{\phi''(x)}{\phi(x)} + \frac{w''(y)}{w(y)} = 0$$

$$\frac{w''(y)}{w(y)} = -\frac{\phi''(x)}{\phi(x)} = \lambda$$

constant since left depends only on y and right depends only on x

This leads to the eigenvalue-eigenfunction problems

next this one

$$\boxed{\begin{aligned} & w''(y) = \lambda w(y) \\ & w(0) = u(x,0) = 0 \\ & w(H) = u(x,H) = f_2(x) \end{aligned}}$$

$$\boxed{\begin{aligned} & \phi''(x) = -\lambda \phi(x) \\ & \phi(0) = u(0,y) = 0 \\ & \phi(W) = u(W,y) = 0 \end{aligned}}$$

do the homogeneous one first...

since bnd. in homogeneous these eigen funcs turn out to be orthogonal

General solution: $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$\phi(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 = 0$

$\phi(W) = c_1 \sin(\sqrt{\lambda}\, W) = 0$

can't have $c_1 = 0$ since then the whole function would be zero and zero is not an eigenfunction...

Thus, $\sin(\sqrt{\lambda}\, W) = 0$ $\quad \sqrt{\lambda}\, W = \pi n$ where n is an integer.

$$\lambda = \left(\frac{\pi n}{W}\right)^2$$

$w''(y) = \lambda w(y)$ $\qquad w(0) = 0$ $\qquad \cancel{w(H) = f_2(x)}$

$\lambda = \left(\frac{n\pi}{W}\right)^2$

use superposition to create a sum that satisfies the B.C. later...

note the general solution could also have been written as a combination of hyperbolic functions

$$w''(y) = \left(\frac{n\pi}{W}\right)^2 w(y)$$

General solution $\quad w(y) = b_1 e^{\frac{n\pi}{W}y} + b_2 e^{-\frac{n\pi}{W}y}$

$$w(0) = b_1 e^{0} + b_2 e^{-0} = 0 \qquad b_1 = -b_2$$

$$w(y) = 2b_1 \left(\frac{e^{\frac{n\pi}{W}y} - e^{-\frac{n\pi}{W}y}}{2}\right)$$

let $a_1 = 2b_1$ ... $\qquad = 2b_1 \sinh\left(\frac{n\pi}{W}y\right) = a_1 \sinh\left(\frac{n\pi}{W}y\right)$

Therefore $\quad u(x,y) = \phi(x)\, w(y) = a_1 \sinh\left(\frac{n\pi}{W}y\right) \sin\left(\frac{n\pi}{W}x\right)$

Now choose constants a_n so that...

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{W}y\right) \sin\left(\frac{n\pi}{W}x\right)$$

Satisfies $\quad u(x,W) = f_2(x)$

How? use orthogonality of the ϕ eigenfunction...

$$\sum_{n=1}^{\infty} a_n \sinh(n\pi) \sin\left(\frac{n\pi}{W}x\right) = f_2(x)$$

$$\int_0^W \sum_{n=1}^{\infty} a_n \sinh(n\pi) \sin\left(\frac{n\pi}{W}x\right) \sin\left(\frac{m\pi}{W}x\right) dx = \int_0^W f_2(x) \sin\left(\frac{m\pi}{W}x\right) dx$$

orthogonality $\qquad \int_0^W \sin\left(\frac{n\pi}{W}x\right) \sin\left(\frac{m\pi}{W}x\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{W}{2} & \text{if } n = m \end{cases}$

Thus

$$a_m \sinh(m\pi)\ \frac{W}{2} = \int_0^W f_2(x) \sin\left(\frac{m\pi}{W}x\right) dx$$

and

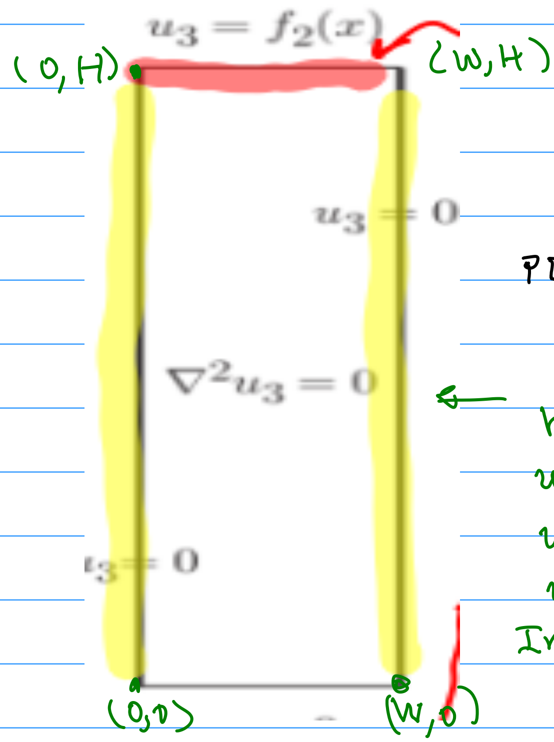
$$a_m = \frac{2}{W \sinh(m\pi)} \int_0^W f_2(x) \sin\left(\frac{m\pi}{W}x\right) dx$$

Therefore

$$u_3(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{W}y\right) \sin\left(\frac{n\pi}{W}x\right)$$

$$\text{where} \quad a_m = \frac{2}{W \sinh(m\pi)} \int_0^W f_2(x) \sin\left(\frac{m\pi}{W}x\right) dx$$

Satisfies the problem



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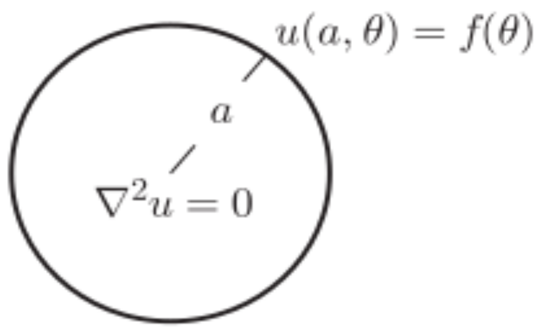
One more example in 2.5...

on of θ, $u(a,\theta) = f(\theta)$. The problem we want to solve

$$\text{PDE:} \quad \boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

$$\text{BC:} \quad \boxed{u(a,\theta) = f(\theta).}$$

ace's equation
lisk.



really there is another boundary condition:

$$u(r,0) = u(r,2\pi)$$

periodic boundary condition in the θ direction like the 1-D heat equation on a ring that we did before.

Use superposition and separation of variables

because

$$u(r,\theta) = \phi(\theta)\,G(r)$$

ϕ represents eigenfunctions that will be orthogonal ...

plug it in ...

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\phi(\theta)G(r)\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\phi(\theta)\,G(r)\right) = 0$$

$$\frac{1}{r}\phi(\theta)\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) + \frac{1}{r^2}G(r)\frac{d^2}{d\theta^2}\phi(\theta) = 0$$

$$\frac{1}{r}\phi(\theta)\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = -\frac{1}{r^2}G(r)\frac{d^2}{d\theta^2}\phi(\theta)$$

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right)}{\frac{1}{r^2}G(r)} = -\frac{\frac{d^2}{d\theta^2}\phi(\theta)}{\phi(\theta)} = \lambda$$

only a function of r only a function of θ

Two eigenvalue eigenfunction problems:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = \lambda\frac{1}{r^2}G(r)$$

this is Euler ODE also called a Cauchy or equidimensional ODE... it can be solved...

$$\underbrace{\frac{d^2}{d\theta^2}\phi(\theta) = -\lambda\phi(\theta)}_{}$$

periodic bndry... we get orthogonality.