- HW2 will be announced soon check tonight or tomorrow
- Solutions to the Quiz soon

Solving Laplace equation $\nabla^2 u = 0$ on the disc...

$$\text{PDE:} \quad \boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

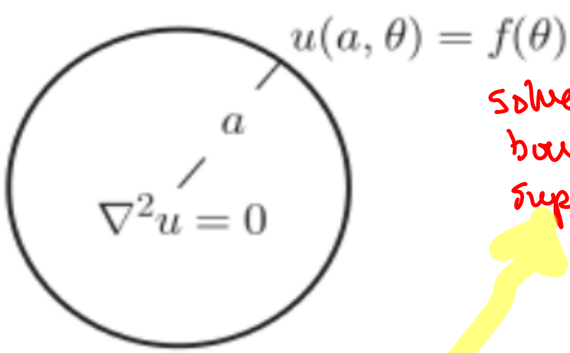
use polar coordinates

$$\text{BC:} \quad \boxed{u(a,\theta) = f(\theta).} \qquad \theta \in [-\pi, \pi]$$

periodicity in θ $\quad u(r,-\pi) = u(r,\pi)$ for $r \in [0,a]$

ce's equation

sk.



solve inhomogeneous boundary using superposition..

use separation of vbls an superposition:

$$u(r,\theta) = \varphi(\theta)\, G(r)$$

Two eigenvalue-eigenfunction problems:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = \lambda \frac{1}{r^2}G(r) \qquad \frac{d^2}{d\theta^2}\varphi(\theta) = -\lambda\,\varphi(\theta)$$

$G(a) = ?$

$G(0) = ?$ maybe $|G(0)| < \infty$

$$\begin{cases} \varphi(-\pi) = \varphi(\pi) \\ \varphi'(-\pi) = \varphi'(\pi) \end{cases}$$

at least physically sensible
actually enough to solve for one of the constants if it were not we'd try something more

First this one:

$$\frac{d^2}{d\theta^2}\varphi(\theta) = -\lambda\varphi(\theta)$$

$$\varphi(-\pi) = \varphi(\pi)$$

General solution

$$\varphi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$\varphi'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

$$\varphi(-\pi) = A\cos(\sqrt{\lambda}\,\pi) - B\sin(\sqrt{\lambda}\,\pi)$$

$$\varphi(\pi) = A\cos(\sqrt{\lambda}\,\pi) + B\sin(\sqrt{\lambda}\,\pi)$$

Thus. $A(\cos(\sqrt{\lambda}\,2\pi) - 1) + B\sin(\sqrt{\lambda}\,2\pi) = 0$

A goes w/ cosine

$$\cancel{A\cos(\sqrt{\lambda}\,\pi)} - B\sin(\sqrt{\lambda}\,\pi) = \cancel{A\cos(\sqrt{\lambda}\,\pi)} + B\sin(\sqrt{\lambda}\,\pi)$$

$$2B\sin(\sqrt{\lambda}\,\pi) = 0$$

either $B = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$

$\sqrt{\lambda} = n$ where $n \in \mathbb{Z}$

$\lambda = n^2$

$$\varphi'(-\pi) = An\sin(\sqrt{\lambda}\,\pi) + Bn\cos(\sqrt{\lambda}\,\pi)$$

$$\varphi'(\pi) = -An\sin(\sqrt{\lambda}\,\pi) + Bn\cos(\sqrt{\lambda}\,\pi)$$

$$A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$$

$$2A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0$$

either $A=0$ or $\sin(\sqrt{\lambda}\pi)=0$

$\sqrt{\lambda}=n$ where $n \in \mathbb{Z}$

$\lambda = n^2$

$\big(B=0$ or $\sin(\sqrt{\lambda}\pi)=0\big)$ and $\big(A=0$ or $\sin(\sqrt{\lambda}\pi)=0\big)$

if $\sin(\sqrt{\lambda}\pi) \neq 0$ then both $A=0$ and $B=0$ which leads to the zero function... Thus we know that $\boxed{\sqrt{\lambda}=n}$ is required...

The other ODE

$$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = \lambda G(r)$$

Euler's equation..

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) = \lambda\frac{1}{r^2}G(r)$$

$|G(0)| < \infty$

$\lambda = n^2$ here

Guess and check method to solve this

Plug in $G(r) = r^p$ and solve for p. Thus,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}r^p\right) = n^2\frac{1}{r^2}r^p$$

$$\frac{1}{r}\frac{d}{dr}\left(r\,p\,r^{p-1}\right) = n^2 r^{p-2}$$

$$\frac{p}{r}\frac{d}{dr}\left(r^p\right) = n^2 r^{p-2}$$

$$\frac{p}{r}\, p r^{p-1} = n^2 r^{p-2}$$

$$p^2 \cancel{r^{p-2}} = n^2 \cancel{r^{p-2}}$$

Solve for p to get $p^2 = n^2$ or $p = \pm n$.

General solution $\boxed{G(r) = c_1 r^n + c_2 r^{-n}}$

this term is not finite as $r \to 0$
$|G(0)| < \infty$
implies $c_2 = 0$

Therefore...

$$G(r) = c_1 r^n$$

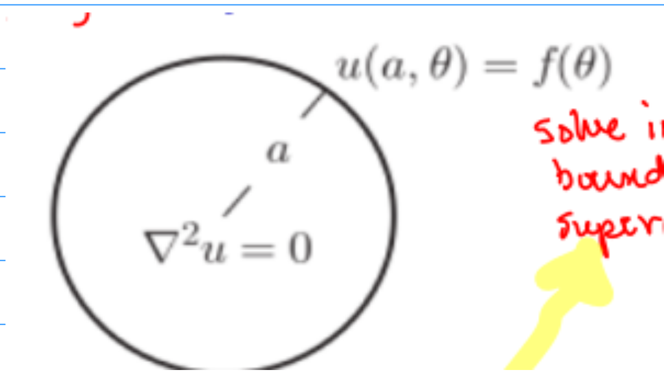
Supperposition:

more sensible not to swap a's and b's here...

$$u(r,\theta) = \sum_{n=0}^{\infty} \left\{ a_n r^n \sin(n\theta) + b_n r^n \cos(n\theta) \right\}$$

$$= b_0 + \sum_{n=1}^{\infty} \left\{ a_n r^n \sin(n\theta) + b_n r^n \cos(n\theta) \right\}$$

Satisfy the boundary condition



$$u(a,\theta) = b_0 + \sum_{n=1}^{\infty}\left\{a_n a^n \sin(n\theta) + b_n a^n \cos(n\theta)\right\} = f(\theta)$$

solve for the constants using orthogonality...

$$\int_{-\pi}^{\pi}\left(b_0 + \sum_{n=1}^{\infty}\left\{a_n a^n \sin(n\theta) + b_n a^n \cos(n\theta)\right\}\right)\sin m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

Thus by orthogonality

$$\int_{-\pi}^{\pi} a_m a^m \underbrace{\sin^2(m\theta)\, d\theta}_{\text{integral is } 1/2} = \int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

$$\frac{\cancel{2}\pi}{\cancel{2}} a_m a^m = \int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

for $m \geq 1$

$$a_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

$$b_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} f(\theta)\cos m\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \left\{ b_0 + \sum_{n=1}^{\infty} \left\{ \cancel{a_n a^n \sin(n\theta)} + \cancel{b_n a^n \cos(n\theta)} \right\} \cdot 1 \right\} d\theta = \int_{-\pi}^{\pi} f(\theta) \cdot 1 \, d\theta$$

$$\int_{-\pi}^{\pi} b_0 \, d\theta = \int_{-\pi}^{\pi} f(\theta) \, d\theta$$

$$2\pi b_0 = \int_{-\pi}^{\pi} f(\theta) \, d\theta$$

Thus,
$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta$$ average value for f.

Solution is

$$u(r, \theta) = b_0 + \sum_{n=1}^{\infty} \left\{ a_n r^n \sin(n\theta) + b_n r^n \cos(n\theta) \right\}$$

where $$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta$$

and for $m \geq 1$ $$a_m = \frac{1}{\pi a^m} \int_{-\pi}^{\pi} f(\theta) \sin m\theta \, d\theta$$

$$b_m = \frac{1}{\pi a^m} \int_{-\pi}^{\pi} f(\theta) \cos m\theta \, d\theta$$

Solution is

$$u(r,\theta) = b_0 + \sum_{n=1}^{\infty}\left\{a_n r^n \sin(n\theta) + b_n r^n \cos(n\theta)\right\}$$

where $b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$

and for $m \geq 1$ $a_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} f(\theta) \sin m\theta \, d\theta$

$$b_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} f(\theta) \cos m\theta \, d\theta$$

oops... swapped A_n with b_n and B_n with a_n in the lecture....

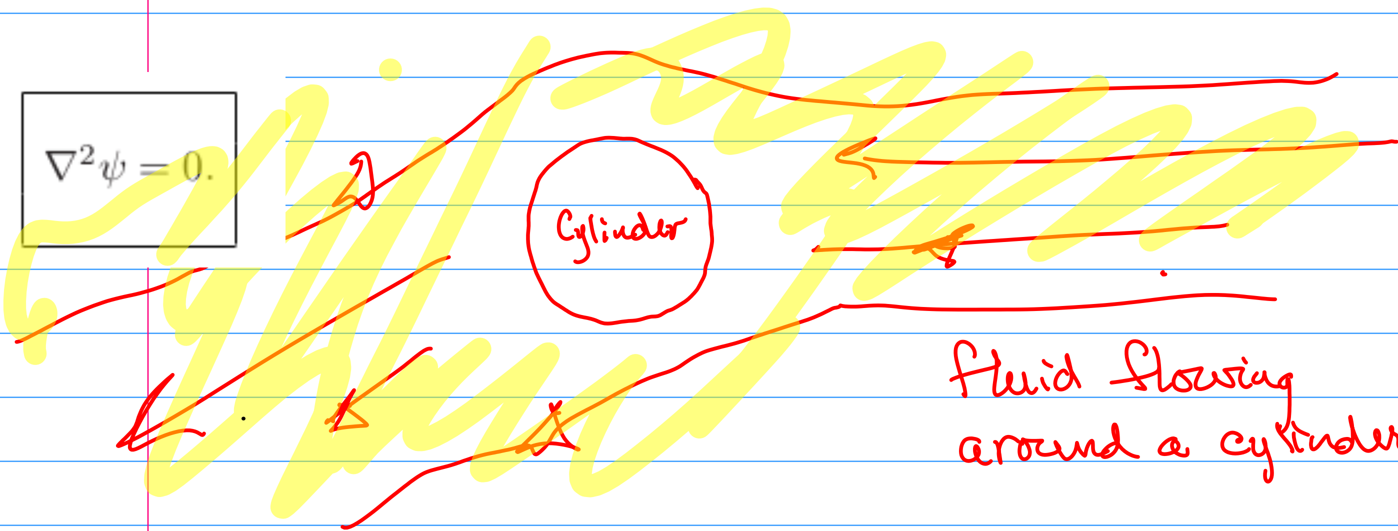
$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta,}$$

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ d\theta \\ (n \geq 1) \quad A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta\ d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta\ d\theta. \end{aligned}$$

Last example in section 2.5.3...

2.5.3 Fluid Flow Outside a Circular Cylinder (Lift)

$$\boxed{\nabla^2\psi = 0.}$$



fluid flowing around a cylinder...

Domain is everything outside the cylinder

General solution...

$$\psi(r,\theta) = c_2 + c_1 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \sin n\theta,$$

how to solve for the constants..., assume a form for ψ as $r \to \infty$ that looks like

$$\boxed{\psi \approx Uy = Ur\sin\theta \quad \text{for large } r.}$$

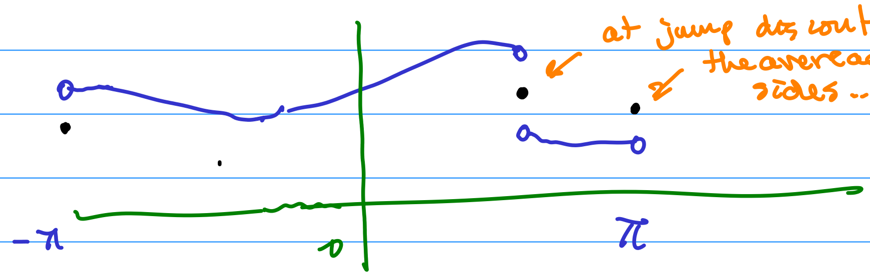
To obtain

$$\boxed{\psi(r,\theta) = c_1 \ln\frac{r}{a} + U\left(r - \frac{a^2}{r}\right)\sin\theta.}$$

Please read this and do the alg. over the weekend...

Convergence theorem for Fourier series. At first, we state a theorem summarizing certain properties of Fourier series:

> If $f(x)$ is *piecewise smooth* on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges
> 1. to the *periodic extension* of $f(x)$, where the periodic extension is continuous;
> 2. to the average of the two limits, usually
> $$\frac{1}{2}\left[f(x+) + f(x-)\right],$$
> where the periodic extension has a *jump discontinuity*.



at jump discontinuity is the avereage of both sides ...

$$f(x) \simeq a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

except at the jump discontinuities