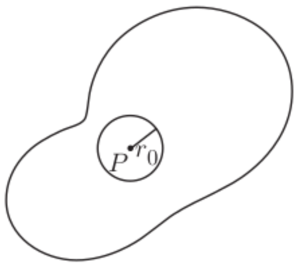
One last observation from chapter two:

Mean value theorem. Our solution of Laplace's equation inside a circle, obtained in Section 2.5.2 by the method of separation of variables, yields an important result. If we evaluate the temperature at the origin, $r = 0$, we discover from (2.5.45) that

$$u(0, \theta) = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta;$$

at any point is the average of the temperature along any circle of radius r_0 (*lying inside* R) centered at that point.

FIGURE 2.5.4 Circle within any region.



$\Delta u = 0$

please read Section 2.5.4 over the weekend

Chapter 3: Fourier Series

1. Separation of variables
2. Homogeneous boundary condition..
3. Self-adjoint differential operator.
4. eigenbasis of orthogonal functions...

for heat equation sines and cosines

$$
\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\; dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\; dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\; dx.
\end{aligned}
$$

$$
f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
$$

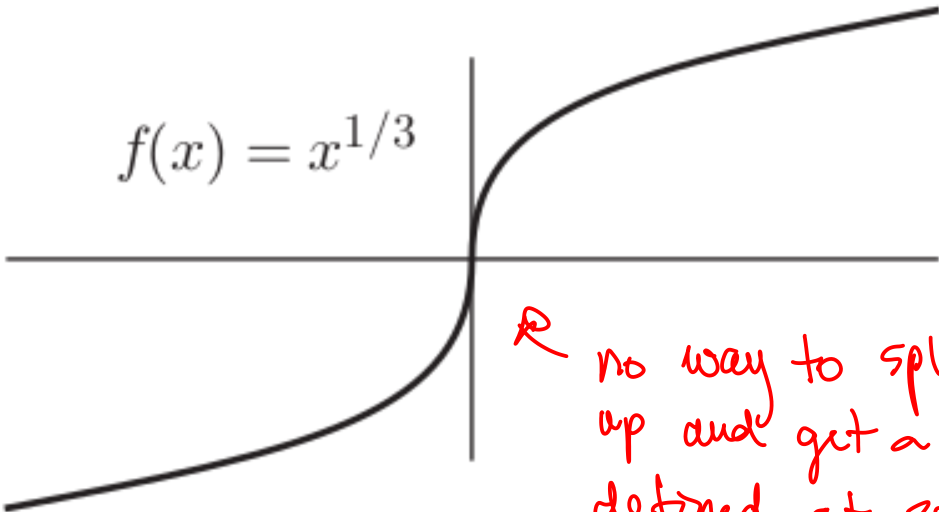
If $f(x)$ is *piecewise smooth* on the interval $-L \leq x \leq L$, then the Fourier series of $f(x)$ converges

1. to the *periodic extension* of $f(x)$, where the periodic extension is continuous;
2. to the average of the two limits, usually

$$
\frac{1}{2}\left[f(x+) + f(x-)\right],
$$

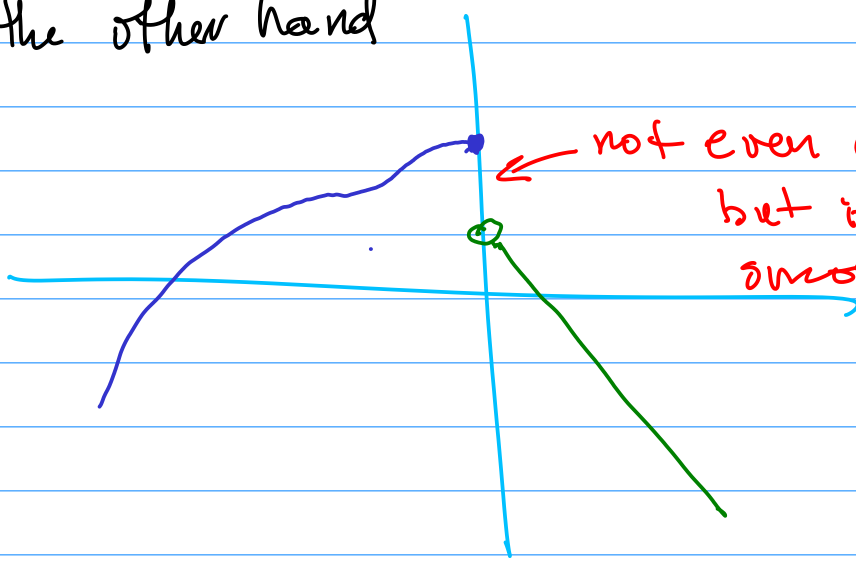
where the periodic extension has a *jump discontinuity.*

This function is not piecewise smooth...



no way to split this up and get a derivative defined at zero

On the other hand



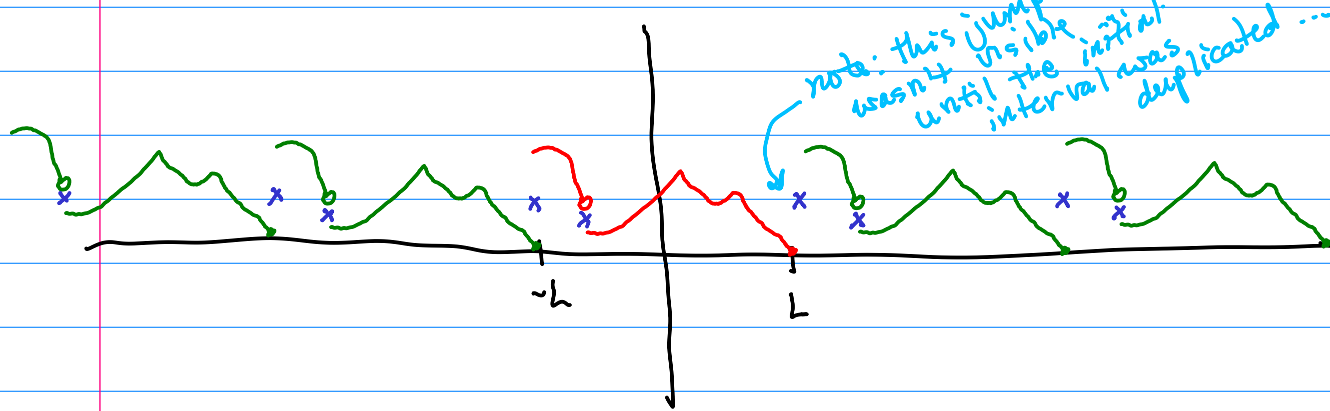
not even continuous...
but it's piecewise smooth

the blue piece is a smooth function
the green piece is a smooth function
So join them together and you have piecewise smooth ...

Sketching Fourier Series of a piecewise smooth function (using the convergence theorem)

1. All series converge (if they converge) (they do) to a periodic function.
2. Sketch the function on $[-L, L]$.
3. Duplicate it periodically.
4. Fix the points at discontinuity with an X.

Imagine the piecewise smooth function f that's $2L$ periodic



<u>Sine Series</u>: Idea $\displaystyle\int_{-L}^{L} f(x)g(x)\,dx = 0$

if f is even and g is odd
or g is even and f is odd

1 is even functi.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot 1\,dx = 0$$

g even

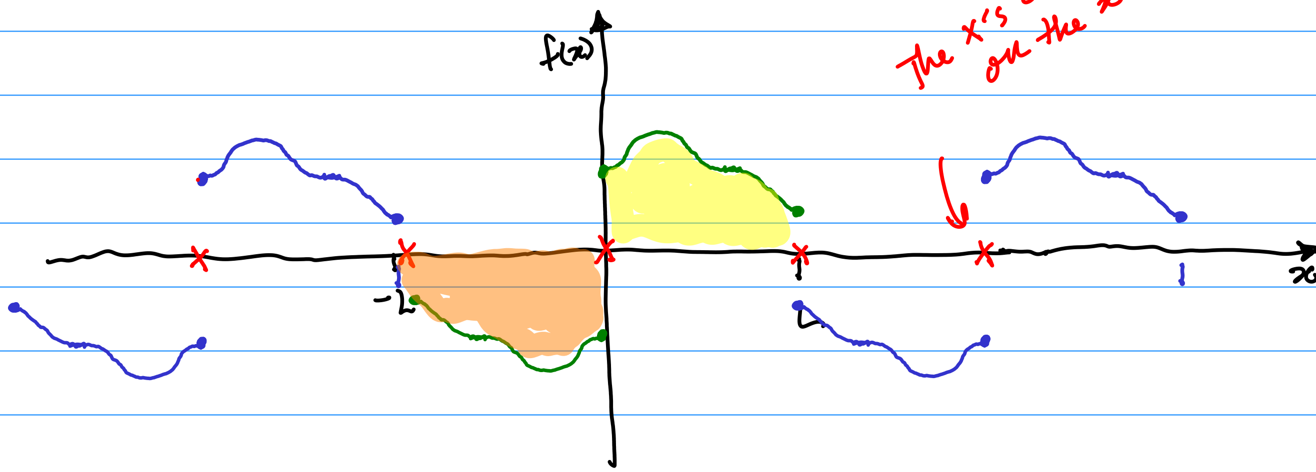
$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{}\,dx = 0 \qquad \text{need } f \text{ to be odd...}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

How to make an odd function?

Start with a function that's piecewise smooth on $[0,L]$,
extend the function by reflection about $(0,0)$ so it's odd
Duplicate periodically

The x's are right on the x-axis.



Now I have an odd function defined that's periodic...
and piecewise smooth...

$$\boxed{f(x) \sim \cancel{a_0} + \cancel{\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},}$$

Thus

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$ by the Fourier conv. theorem.

where

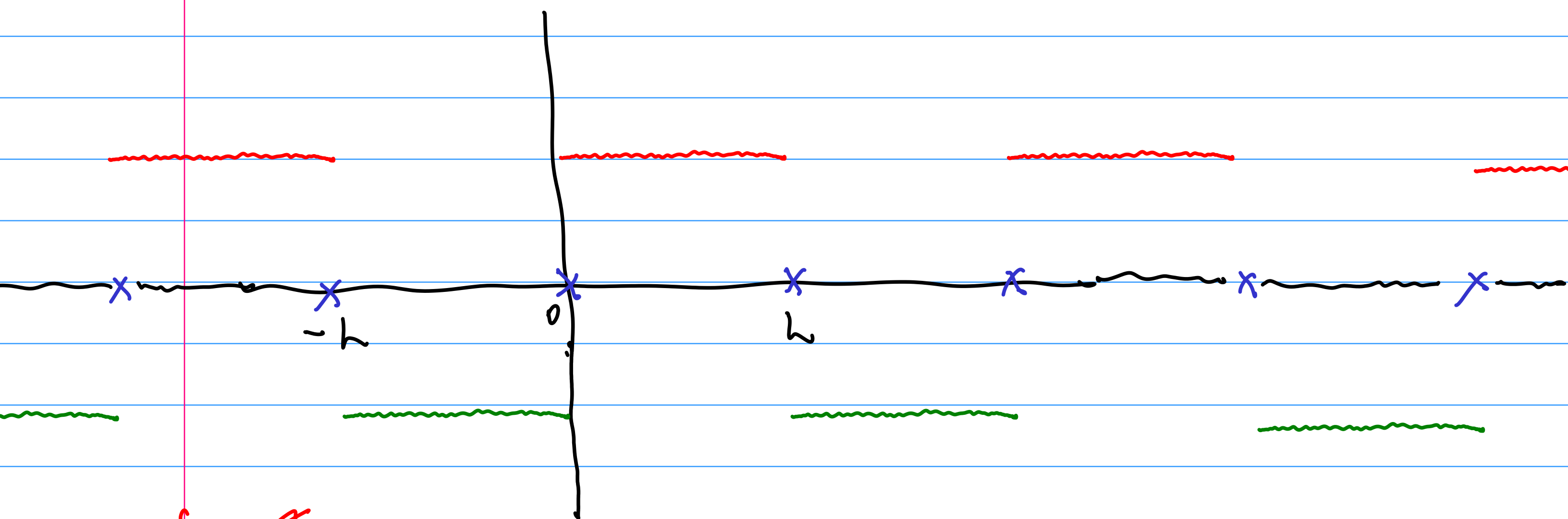
$$b_n = \frac{1}{L}\int_{-L}^{L} \overset{\text{odd}}{f(x)} \overset{\text{odd}}{\sin\frac{n\pi x}{L}}\, dx. = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

This is exactly what we had earlier...

So if $f(x) = 100$ then

$$b_n = \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and



converges to the graph

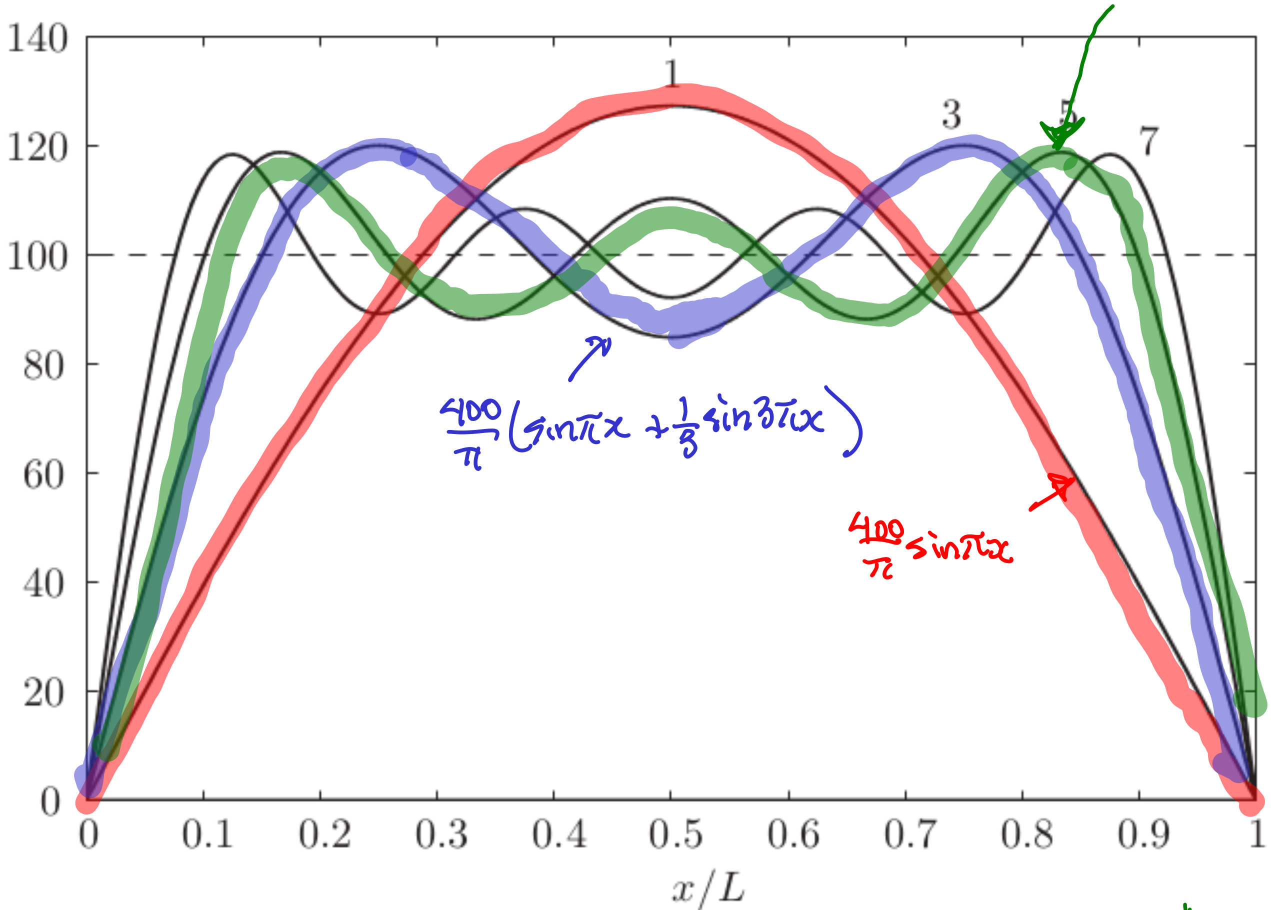
$$\sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{400}{n\pi} \sin \frac{n\pi x}{L}$$

Does the series really look like this? How does it converge? What do finite sums look like?

a series is a limit of larger and larger sums:

$$\lim_{N \to \infty} \sum_{\substack{1 \leq n \leq N \\ n \text{ odd}}} \frac{400}{n\pi} \sin \frac{n\pi x}{L} = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{400}{n\pi} \sin \frac{n\pi x}{L}$$

$\frac{400}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x\right)$

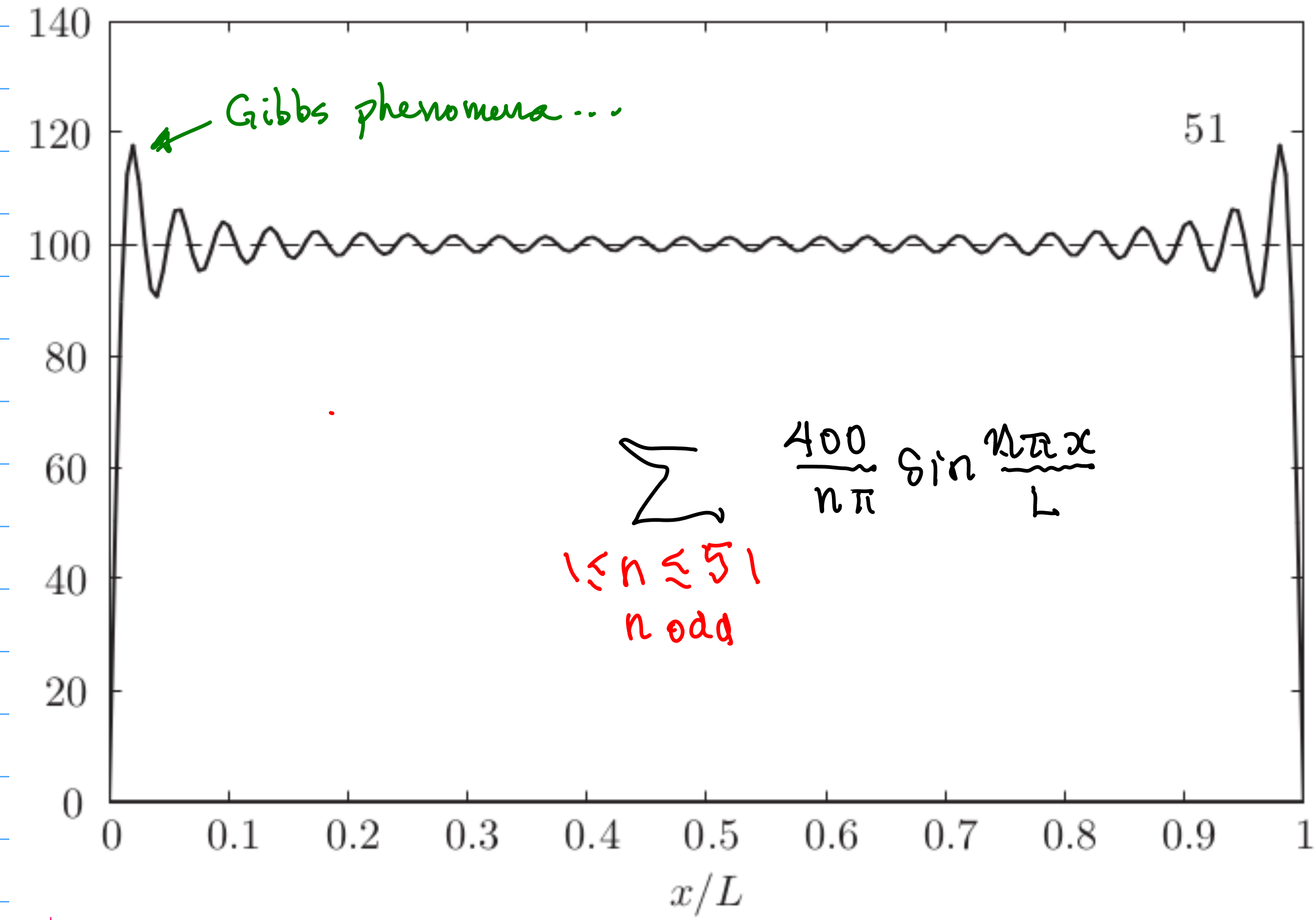


* equivalent to setting L=1 and graphing x

$$\sum_{\substack{1 \le n \le N \\ n \text{ odd}}} \frac{400}{n\pi}\sin\frac{n\pi x}{L}$$

$$= \frac{400}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right)$$

How does this series converges as $N \to \infty$?



$$\sum_{\substack{1 \le n \le 51 \\ n \text{ odd}}} \frac{400}{n\pi} \sin \frac{n\pi x}{L}$$

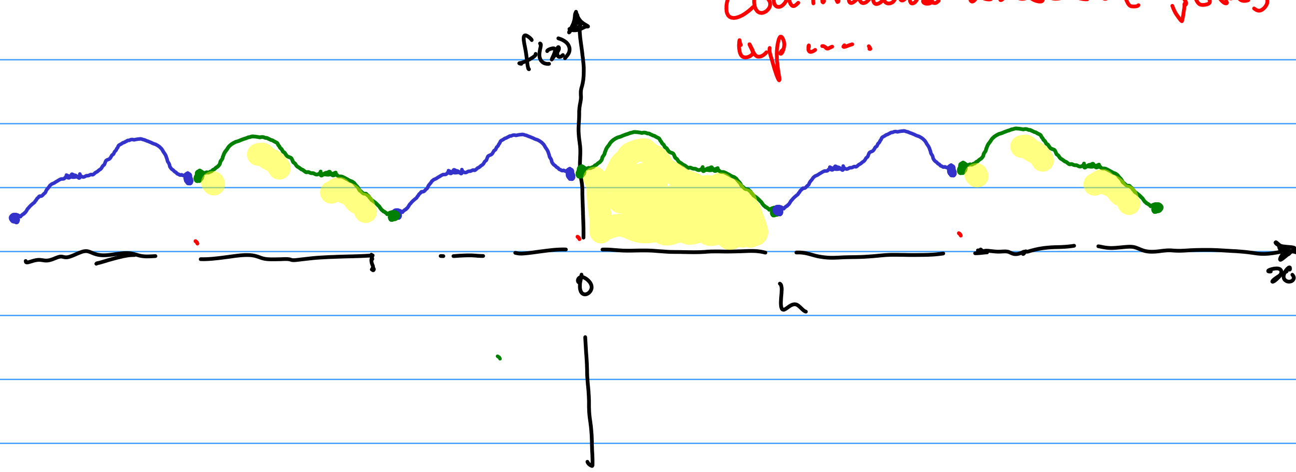
In the limit the spike at the edges gets thinner an thinner, and moves closer to the edges, but does not go away... and is always about 8% of overshoot from what the limit is.

Thus we have pointwise convergence, but not uniform convergence...

Cosine series... You can do something similar to make the sine term go away instead.

Start with a function defined on [0,L]
since sine is odd all the sines go away

note even reflection is continuous where it joins up....



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\,\underbrace{\sin\frac{n\pi x}{L}}_{\text{odd}}\,dx. = 0$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \cancel{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}},$$

whether to use sine or cosine has to do with the boundary condition...