Recall:

Sine series was for odd functions

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad \text{where} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

at points where the odd extension of f is continuous then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

at points of jump discontinuity

$$\frac{f(x^-) + f(x^+)}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

note x^- and x^- are with respect to the function that's been oddly extended.

Cosine series for even functions...

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad \text{where} \qquad a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$\text{and } a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Idea: write any function that is periodic with period $2L$ as a sum of even and an odd function:

$$f(x) = \frac{1}{2}\Big(\underbrace{f(x) + f(-x)}_{\text{even since switch } x \text{ for } -x \text{ and nothing happens}}\Big) + \frac{1}{2}\Big(\underbrace{f(x) - f(-x)}_{\text{odd since switch } x \text{ for } -x \text{ and it changes sign.}}\Big)$$

$$f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

Then $f(x) = f_e(x) + f_o(x)$

suppose $f(x)$ is piecewise smooth and continuous ...

thus f_e and f_o are also piecewise smooth and cont.

$$f_e(x) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{where} \quad a_0 = \frac{1}{L}\int_0^L f_e(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f_e(x)\cos\frac{n\pi x}{L}\,dx$$

$$f_o(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f_o(x)\sin\frac{n\pi x}{L}\,dx$$

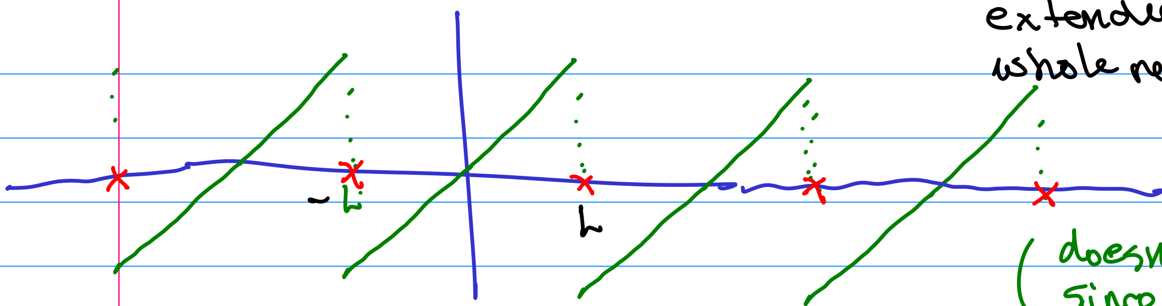
Therefore,

$$f(x) \approx f_e(x) + f_o(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Differentiation of Fourier series

- It doesn't always work.

Example $f(x) = x$ on $[-L, L]$ periodically extended on the whole real line



(doesn't work since $f(L) \neq 0$)

Find the Fourier series of f: Since $f(x)$ is odd I can use a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx$$

by parts $\quad u = x \qquad du = dx$

$\qquad dv = \sin \frac{n\pi x}{L} \qquad v = \frac{-L}{n\pi} \cos \frac{n\pi x}{L}$

$$= \frac{2}{L} \left(x \left(\frac{-L}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_0^L - \cancel{\int_0^L \left(\frac{-L}{n\pi} \cos \frac{n\pi x}{L} \right) dx} \right)$$

$$= \frac{-2}{n\pi} L \cos n\pi = \frac{-2L}{n\pi} (-1)^n = \frac{2L}{n\pi} (-1)^{n+1}$$

Thus

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}$$

Moreover sine $f(x)$ is continuous on $(-L, L)$ thus

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L} \quad \text{for} \quad x \in (-L, L)$$

If it were possible to differentiate this Fourier series term by term (it's not) then

$$1 = \sum_{n=1}^{\infty} \frac{2\cancel{L}}{\cancel{n}\cancel{\pi}}(-1)^{n+1}\left(\cos\frac{n\pi x}{L}\right)\frac{\cancel{n}\cancel{\pi}}{\cancel{L}} \qquad \text{(wrong)}$$

Thus,

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos\frac{n\pi x}{L} \qquad x \in (-L, L)$$

(wrong)

Plug in $x=0$ and

(obviously wrong)

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} = 2 - 2 + 2 - 2 + 2 - 2 + 2 - \cdots$$

This is not even a convergent series... definitely not equal to 1.

Conclusion: It's not possible to differentiate all Fourier series term by term.

Question: When is it possible?

A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

means $f'(x)$ and $f''(x)$ are piecewise continuous functions...

Typo...

If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

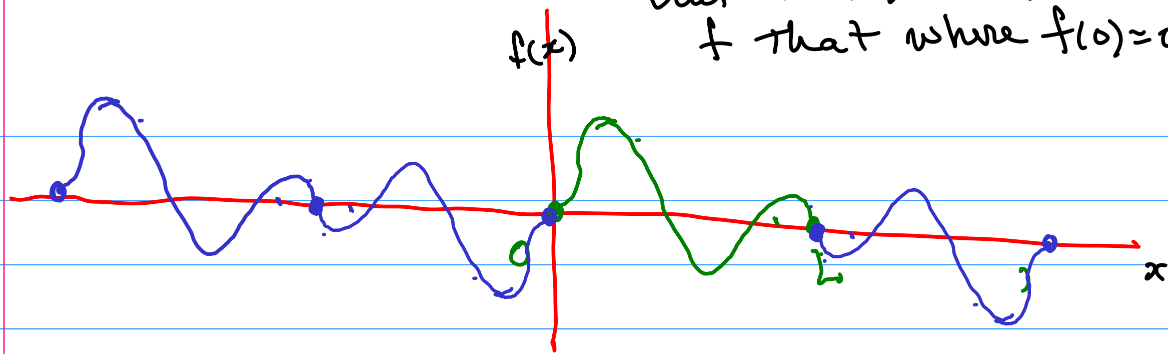
If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term.

If $f'(x)$ is piecewise smooth, then a continuous Fourier sine series of $f(x)$ can be differentiated term by term.

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

so far " ⟸ " of the implicatio

odd extension of continuous f that where f(0)=0, f(L)=0



Since it's continuous, The Fourier conv. theorem says

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

The derivative $f'(x)$ is piecewise smooth... (but might not be continuous). Where it's continuous we can write the Fourier series of $f'(x)$...

Since f' is derivative of an odd function then f' must be even...

Thus I can write $f'(x)$ as a cosine series

$$f'(x)=a_0+\sum_{n=1}^{\infty} \boxed{a_n} \cos\frac{n\pi x}{L} \qquad a_0=\frac{1}{L}\int_0^L f'(x)\,dx$$

since piecewise smooth where $f'(x)$ is continuous

$$a_n=\frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx$$

Trying to see why term by term differentiation works... Thus it's enough to differentiate

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Term by term and check its the same series as I get directly for f' using the conv. theorem

$$\frac{d}{dx}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} \boxed{b_n \frac{n\pi}{L}} \cos\frac{n\pi x}{L}$$

need to show $b_n \frac{n\pi}{L} = a_n$ where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$a_0 = \frac{1}{L}\int_0^L f'(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx$$

Since there isn't a b_0 at all, then we need to show that $a_0 = 0$ ☑

$$a_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\left(f(L) - f(0)\right) = \frac{1}{L}(0-0) = 0$$

since, $f(0)=0$, $f(L)=0$

$$a_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx =$$

this function has to be continuous...

when can you really do integration by parts?

$dv = f'(x)\,dx$ $\quad v = f(x)$

$u = \cos\frac{n\pi x}{L}$ $\quad du = -\frac{n\pi}{L}\sin\frac{n\pi x}{L}$

$$= \frac{2}{L}\left(f(x)\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L f(x)\left(-\frac{n\pi}{L}\sin\frac{n\pi x}{L}\right)dx \right)$$

$$= \frac{2}{L}\left(\cancel{f(L)\cos} n\pi - \cancel{f(0)} \right) + \frac{n\pi}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx)$$

$$= \frac{n\pi}{L}\,\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

recall

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore

$$a_n = \frac{n\pi}{L}\,b_n$$

That's what I wanted...

$$b_n\frac{n\pi}{L} = a_n$$