Dear Class, Somehow I deleted the lecture notes from Tuesday, March 14.
I have tried to recreate the entire lecture as best as I remember.

Announcements:

I posted my solutions to HW2.

I posted a sample exam for the Midterm.

The midterm will be in-class Thursday March 16.

Outline for today:

Discuss the integration of Fourier series.

Discuss the sample exam.

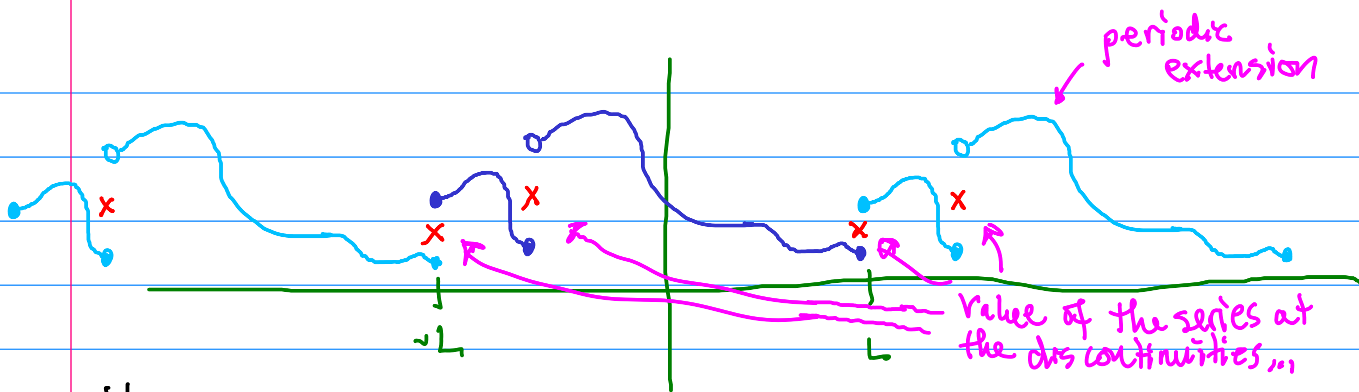
Integration of Fourier Series:

> A Fourier series of piecewise smooth $f(x)$ can always be integrated term by term, and the result is a *convergent* infinite series that always *converges* to the integral of $f(x)$ for $-L \leq x \leq L$ (even if the original Fourier series has jump discontinuities).

Note the theorem is simpler that the differentiation theorems. This is because integration has an averaging effect that makes the resulting function smoother.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Since f is piecewise smooth the Fourier series converges at each point of continuity of the $2L$-periodically extended function and to the average value at the jumps.



Here

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

The theorem says we can integrate term by term and obtain exact inequality even at the jumps. Thus

$$\int_{-L}^{x} f(s)\,ds \approx a_0(x+L) + \sum_{n=1}^{\infty} a_n \int_{-L}^{x} \cos\frac{n\pi s}{L}\,ds + \sum_{n=1}^{\infty} b_n \int_{-L}^{x} \sin\frac{n\pi s}{L}\,ds$$

holds for all $x \in [-L, L]$

Integrate,

$$\int_{-L}^{x} \cos\frac{n\pi s}{L}\,ds = \frac{L}{n\pi}\sin\frac{n\pi s}{L}\Big|_{-L}^{x} = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$\int_{-L}^{x} \sin\frac{n\pi s}{L}\,ds = -\frac{L}{n\pi}\cos\frac{n\pi s}{L}\Big|_{-L}^{x} = \frac{-L}{n\pi}\cos\frac{n\pi x}{L} + \frac{L}{n\pi}\cos n\pi$$

this term does not depend on x

$$= \frac{-L}{n\pi}\cos\frac{n\pi x}{L} + \frac{L}{n\pi}(-1)^n$$

Plug it back in to obtain

$$\int_{-L}^{x} f(s)\,ds \approx \underbrace{a_0(x+L) + \sum_{n=1}^{\infty} b_n \frac{L}{n\pi}(-1)^n}_{\text{constant term}} + \sum_{n=1}^{\infty} b_n \left(\frac{-L}{n\pi}\right)\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right)\sin\frac{n\pi x}{L}$$

which holds for all $x \in [-L, L]$.

note the n in the denominator makes the series converge faster, which is good.

Now let's look at the sample exam...

3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in [0, L]$$

subject to the homogeneous boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = 5\sin\left(\frac{3\pi x}{2L}\right).$$

One could assume $\sin\left(\frac{3\pi x}{2L}\right)$ is an eigenfunction for the homogeneous direction and guess the solution

$$u(x,t) = 5\sin\frac{3\pi x}{2L}\, e^{-\left(\frac{3\pi}{2L}\right)^2 t}$$

error here k is missing

Even if the guess were correct, it does not communicate well how to solve the problem.

Solving is different than writing down the solution

So from the beginning...

For separation of variables write $u(x,t) = \phi(x) g(t)$. Then

$$\phi(x) g'(t) = k\,\phi''(x) g(t)$$

or

$$\frac{g'(t)}{k g(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

On one hand λ doesn't depend on x
on the other λ doesn't depend on t
Therefore it is constant.

We obtain the ODEs

$$g'(t) = -\lambda k g(t) \qquad \text{and} \qquad \phi''(x) = -\lambda \phi(x)$$

with general solutions

assuming $\lambda > 0$

$$g(t) = c e^{-\lambda k t} \qquad \text{and} \qquad \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$\phi'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

The boundary conditions

the cases $\lambda = 0$ and $\lambda < 0$ would have to be checked separately. It turns out $\lambda > 0$ is enough..

$$u(0,t) = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

translate to $\phi(0) = 0$ and $\phi'(L) = 0$.

- Note that these boundary conditions are homogeneous. Thus, the sum of solutions that satisfy the boundary conditions again satisfy the boundary conditions. This is important for superposition.
- Homogeneous boundary conditions will also imply the resulting eigenfunctions are orthogonal.

Solving for the boundary conditions

$$\varphi(0) = A\cos\sqrt{\lambda}\,0 + B\sin\sqrt{\lambda}\,0 = A = 0 \qquad \text{so } A = 0$$

$$\varphi'(L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}L + B\sqrt{\lambda}\cos\sqrt{\lambda}L = B\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$

since $A=0$

If $B=0$ then $\varphi(x)=0$, but that would not be an eigenfunction at all.

Therefore $B \neq 0$, in which case $\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$.

$\lambda > 0$ so $\cos\sqrt{\lambda}L = 0$

$$\sqrt{\lambda}L = \frac{\pi}{2} + n\pi \quad \text{for } n = 0, 1, 2, \ldots$$

Note $\sqrt{\lambda} = \left(\frac{\pi}{2} + n\pi\right)\frac{1}{L}$ and $\lambda = \left(\frac{\pi}{2} + n\pi\right)^2 \frac{1}{L^2}$

Superposition now leads to the solution

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right) e^{-\left[\left(\frac{\pi}{2}+n\pi\right)/L\right]^2 kt}$$

The red letters indicate the places errors were corrected in class. It's important to check for errors during the solution of a problem, because errors not only lead to wrong answers but can make the entire calculation more difficult. For example, if the sin were mistakenly replaced by cos then identifying the B_n in the next step is much more difficult.

Now solve for the B_n's to realize the initial condition

$$u(x,0) = 5\sin\left(\frac{3\pi x}{2L}\right).$$

Since

$$\sin\left(\frac{3\pi x}{2L}\right) = \sin\left(\left(\frac{\pi}{2}+\pi\right)\frac{x}{L}\right) = \sin\left(\left(\frac{\pi}{2}+n\pi\right)\frac{x}{L}\right) \quad \text{for } n=1$$

thus we can take $B_1 = 5$ and $B_j = 0$ for $j \neq 1$.

The series then reduces to the single term

$$B_1 \sin\left(\left(\frac{\pi}{2}+1\cdot\pi\right)\frac{x}{L}\right) e^{-\left[\left(\frac{\pi}{2}+1\cdot\pi\right)/L\right]^2 kt} \boxed{= 5\sin\left(\frac{3\pi x}{2L}\right) e^{-\left(\frac{3\pi}{2L}\right)^2 kt}}$$

the answer...

Note that the point is not to write down the answer so much as show how what you know abouts PDE's can be used to find the answer...

When writing your solution consider:

1. I'm not interested in knowing what the solution but instead how to arrive at that answer.
2. Balance details and shortcuts with the time constraint to communicate as much as you can about what you learned in the class.

③ Adding details that are wrong will lead to loss of points.

④ Taking too many shortcuts or not explaining clearly will lead to loss of points.

Suppose I wanted to make the above problem more difficult because it was too easy. What could be done?

① Change the initial condition so its not an eigenfunction. For example

$$u(x,0) = \cos 3x$$

doesn't even satisfy the boundary conditions re $\mathcal{G}$ and would result in an infinite series solution as in homework problem 2.3.3(c).

② Only make it a little more complicated.

$$u(x,0) = 5\sin\frac{3\pi x}{2L} + 7\sin\frac{9\pi x}{2L}$$

In this case the answer would be

$$u(x,t) = 5\sin\frac{3\pi x}{2L}\, e^{-\left(\frac{3\pi}{2L}\right)^2 kt} + 7\sin\frac{9\pi x}{2L}\, e^{-\left(\frac{9\pi}{2L}\right)^2 kt}.$$

Again, sorry I lost the original version of these notes. I think the above turned out pretty close to what we did in class. An improvement in some ways and maybe not in others.

If you see anything I left out, please let me know and I will update these notes.

All the best and good luck on Thursday!