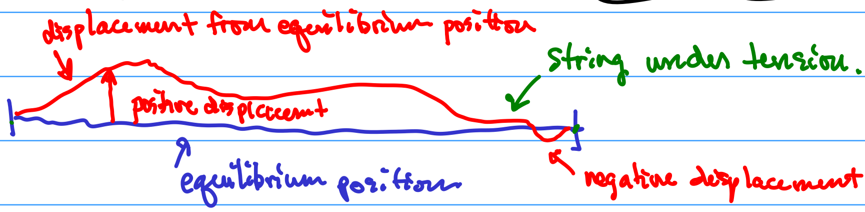
Chapter 4 Back to differential equations...

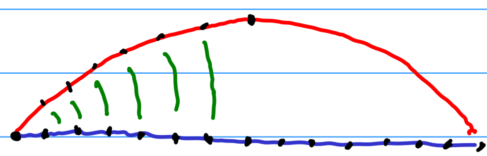
- Time dependent Heat equation — parabolic PDE
  dissipative physics
- Equilibrium solution to 2D Heat equation – Laplace equation and Poisson equation — elliptic PDE
  statics problem
- Wave Equation — hyperbolic PDEs
  conservation law



$u(x,t)$ is the displacement from the equilibrium position

$u>0$ means its above the equilibrium

$u<0$ means its below.



Simplifying assumption the partical trajectories of the black dots occur only in the vertical direction...

This is a good approximation under small amplitudes..

Kinematics problem: mass moving about subject to a force:

Newton's Law: $F = ma$

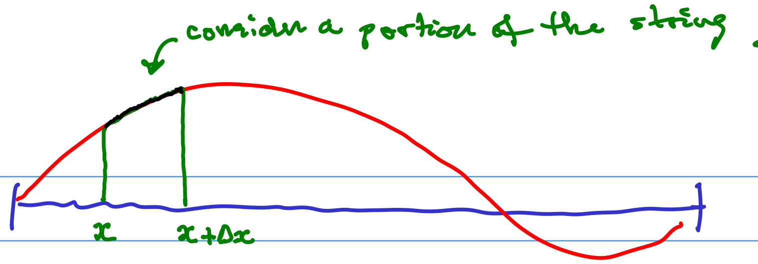
$[u] = [L]$ ← length

acceleration $= a = \dfrac{\partial^2 u}{\partial t^2}$

$[a] = \dfrac{[L]}{[T]^2}$

Assume the string is constant density:
note this is a 1-dimensional density

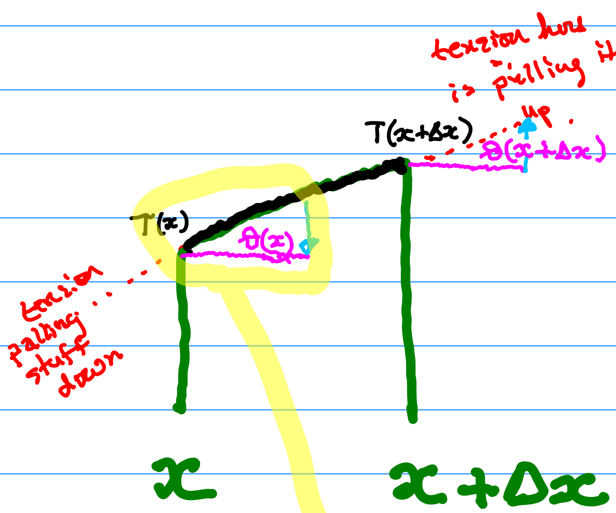
$[\rho_0] = \dfrac{[M]}{[L]}$ ← single power of length.



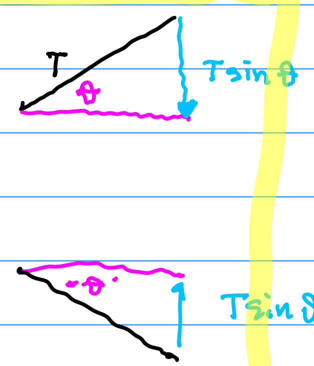
How much mass is in that portion of string $\quad m = \rho_0 \Delta x \qquad \frac{[m]}{\cancel{[L]}}\cancel{[L]} = [m]$

$$F = ma \approx \rho_0 \Delta x \frac{\partial^2 u}{\partial t^2}$$

Now I need the force... comes from the tension in the string



ignore the component of the tension that pulls things left or right because of the simplifying assumption



$$F = T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)$$

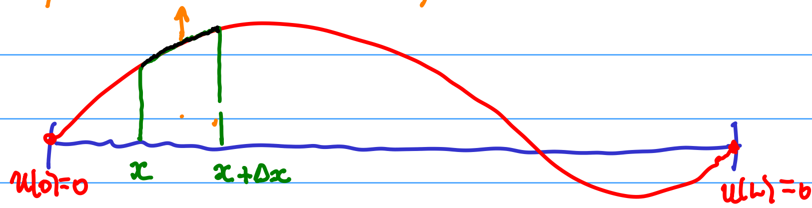
$$F = ma$$

$$\rho_0 \Delta x \frac{\partial^2 u}{\partial t^2} \approx T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)$$

Could be other forces that don't come from the tensions...

(body force)

$\rho_0 \Delta x Q$ extra force pulling this bit of string down or up



note if $\rho_0 \Delta x Q$ is a force then Q is an acceleration ...

Example gravity $Q=-g$

Mathematical model...

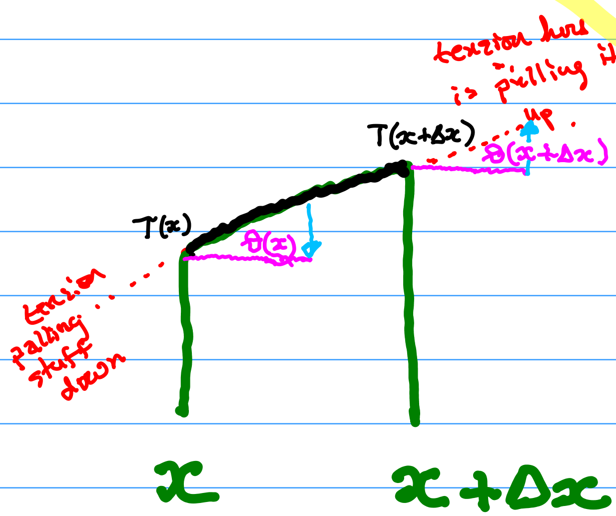
$$\rho_0 \Delta x \frac{\partial^2 u}{\partial t^2} \approx T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x) + \rho_0 \Delta x Q$$

divide by Δx and take limits...

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \underbrace{\frac{T(x+\Delta x)\sin\theta(x+\Delta x) - T(x)\sin\theta(x)}{\Delta x}} + \rho_0 Q$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} T(x) \sin\theta(x) + \rho_0 Q$$

Now we need to substitute for θ in term of u.



Slope of the line $= \frac{\partial u}{\partial x} = \tan\theta$

$= \frac{\sin\theta}{\cos\theta} \approx \sin\theta$

since $\cos\theta \approx 1$ assume θ is small and θ is small by assumption.

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} T(x) \underbrace{\frac{\partial u}{\partial x}} + \rho_0 Q$$

One more assumption: $T(x) = T_0$ is constant

$Q(x) = 0$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

simplest form of wave equation...

let $c^2 = \frac{T_0}{\rho_0}$ ← force, ← 1-d density

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

2nd order in time
need two initial conditions

$$\frac{[T_0]}{[\rho_0]} = \frac{[ma]}{[M]/[L]} = \frac{\cancel{[m]}\,[L]/[T]^2}{\cancel{[m]}/[L]}$$

$$[c]^2 = \frac{[L]^2}{[T]^2}$$

$$[c] = \frac{[L]}{[T]} \quad \text{velocity}$$

Try to solve using superposition principle.

Boundary Conditions:

$u(0,t) = 0$

$u(L,t) = 0$

Initial condition:

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

Physically need to know the position and velocity of each "particle" in the string to predict what it will do next...

Does it just work?

$u(x,t) = \phi(x)\, h(t)$

Satisfies a 2nd order equation, so more constants than before... ok. because there are more initial conditions.

Same differential operator in x and boundary conditions so I expect orthogonality in ϕ (eigenfunctions).

Plug it in

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\phi(x)\, h''(t) = c^2 \phi''(x)\, h(t)$$

$$\frac{h''(t)}{c^2 h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

Thus

$$h''(t) = -\lambda c^2 h(t) \quad \text{and} \quad \boxed{\begin{array}{ll} \varphi''(x) = -\lambda \varphi(x) & \\ \varphi(0)=0 & \varphi(L)=0 \end{array}}$$

$$\varphi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

$$\varphi(0) = A = 0$$

$$\varphi(L) = B\sin\sqrt{\lambda}\,L \qquad \sqrt{\lambda}\,L = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$h(t) = c_1\cos c\sqrt{\lambda}\,t + c_2\sin c\sqrt{\lambda}\,t$$

By super position write

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos c\frac{n\pi}{L}t + b_n\sin c\frac{n\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

now try to solve for a_n and b_n to satisfy the initial condition.
using orthogonality.... Next time.