PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions:

$u(0,t) = 0$

$u(L,t) = 0$

Initial condition:

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

By super position write

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos c\frac{n\pi}{L}t + b_n \sin c\frac{n\pi}{L}t\right) \sin \frac{n\pi}{L}x$$

now try to solve for a_n and b_n to satisfy the initial condition.
using orthogonality.... Next time.

$$u(x,0) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} 0 + b_n \sin \frac{c n\pi}{L} 0\right) \sin \frac{n\pi}{L}x = f(x)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-a_n \frac{cn\pi}{L} \sin \frac{cn\pi}{L} t + b_n \frac{cn\pi}{L} \cos \frac{cn\pi}{L} t\right) \sin \frac{n\pi}{L} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(-a_n \frac{cn\pi}{L} \sin \frac{cn\pi}{L} 0 + b_n \frac{cn\pi}{L} \cos \frac{cn\pi}{L} 0\right) \sin \frac{n\pi}{L} x$$

Thus

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \boxed{\frac{cn\pi}{L}} \sin\frac{n\pi x}{L} = g(x)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{L}{cn\pi}\,\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$[C] = \frac{[L]}{[T]}$ so the L is in c,

In 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

In 2D case (an 3D)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Before that one more observation about 1D.

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos c\frac{n\pi}{L}t + b_n \sin c\frac{n\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

$$= \sum_{n=1}^{\infty} \underbrace{a_n \cos\frac{cn\pi}{L}t\,\sin\frac{n\pi}{L}x} + \underbrace{b_n \sin\frac{cn\pi}{L}t\,\sin\frac{n\pi}{L}x}$$

angle addition formula...

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$\frac{\partial}{\partial\alpha}$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \boxed{\sin\alpha\sin\beta}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \boxed{\sin\alpha\sin\beta}$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

$$\frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} = \sin\alpha\sin\beta$$

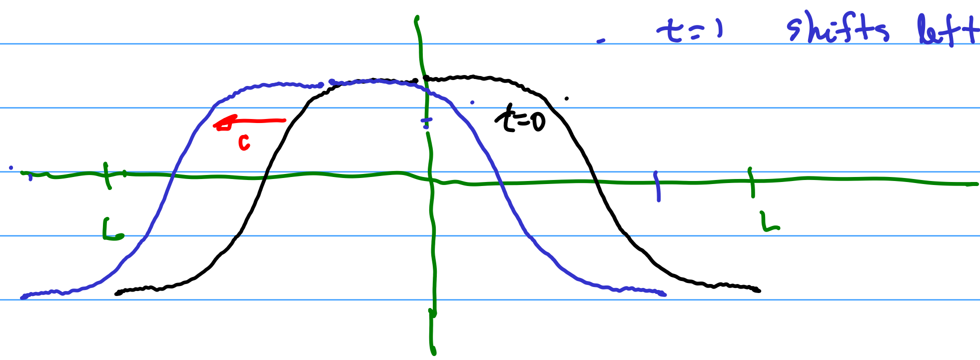
$\cos\left(\frac{-cn\pi}{L}t + \frac{n\pi}{L}x\right)$

$$b_n \sin\frac{cn\pi}{L}t\,\sin\frac{n\pi}{L}x = b_n\frac{\cos\left(\frac{cn\pi}{L}t - \frac{n\pi}{L}x\right) - \cos\left(\frac{cn\pi}{L}t + \frac{n\pi}{L}x\right)}{2}$$

$n=1$ for simplicity

what is $\cos\left((ct+x)\frac{n\pi}{L}\right)$? Traveling wave... to the left

$t=1$ shifts left by how much?



And $\cos\left(\frac{cn\pi}{L}t - \frac{n\pi}{L}x\right)$ is traveling wave to the right...

Subtracting these two traveling waves leads to the standing wave where the boundary conditions at $x=0$ and $x=L$ are satisfied.

Identifying these traveling solution forshadows the method of characteristic later in the course...

Subtract

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{cases}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \boxed{\sin\alpha\sin\beta}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \boxed{\sin\alpha\sin\beta}$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

$$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$$

left with velocity c   right with velocity c...

$$a_n \cos\frac{c n\pi}{L} t \sin\frac{n\pi}{L} x = a_n \frac{\sin\left((x+ct)\frac{n\pi}{L}\right) + \sin\left((x-ct)\frac{n\pi}{L}\right)}{2}$$

Example problems:

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

***(b)** Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

Tension idea:

$$T_0 \frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(T_0\frac{\partial u}{\partial x} - \beta u\right)$$

Velocity idea $\qquad v = \frac{\partial u}{\partial t}$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \cancel{\frac{\partial^2 u}{\partial x^2}} - \beta \frac{\partial u}{\partial t}.$$

$$\rho_0 \frac{\partial v}{\partial t} = -\beta v$$

↑
change in velocity is proportional to $-\beta v$.

since $-\beta$ and $\beta > 0$ then velocity is decreasing...

Try separation of variables

$u(x,t) = \varphi(x) h(t)$ $\qquad$ plug it in...

$$\rho_0 \varphi(x) h''(t) = T_0 \varphi''(x) h(t) - \beta \varphi(x) h'(t)$$

$$\rho_0 h''(t) = T_0 \frac{\varphi''(x)}{\varphi(x)} h(t) - \beta h'(t)$$

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} = \frac{\varphi''(x)}{\varphi(x)} - \frac{\beta}{T_0} \frac{h'(t)}{h(t)}$$

$$\frac{\rho_0}{T_0} \frac{h''(t)}{h(t)} + \frac{\beta}{T_0} \frac{h'(t)}{h(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$$

We obtain

$$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t) \quad \text{and} \quad \varphi''(x) = -\lambda \varphi(x)$$

$$\varphi(0) = 0 \quad \varphi(L) = 0$$

$\varphi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\varphi(0) = A = 0$ $\qquad A = 0$

$\varphi(L) = B\sin\sqrt{\lambda}\,L = 0$ $\qquad \sqrt{\lambda}\,L = n\pi$ $\qquad \sqrt{\lambda} = \frac{n\pi}{L}$

Now solve for $h(t)$ using ...

$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t)$

Plug in e^{rt} to find characteristic equation...

$\rho_0 r^2 e^{rt} + \beta r e^{rt} = -\lambda T_0 e^{rt}$

$\rho_0 r^2 + \beta r + \lambda T_0 = 0$ characteristic equation...

$a = \rho_0 \quad b = \beta \quad c = \lambda T_0$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0\lambda T_0}}{2\rho_0}$$

$\sqrt{\lambda} = \frac{n\pi}{L}$

Thus,

$$4\rho_0\lambda T_0 \geq 4\rho_0 \frac{\pi^2}{L^2} T_0$$

$\lambda = \frac{n^2\pi^2}{L^2} \geq \frac{\pi^2}{L^2}$

and so $\beta^2 - 4\rho_0\lambda T_0 < 0$

$$r = \frac{-\beta \pm i\sqrt{4\rho_0\lambda T_0 - \beta^2}}{2\rho_0}$$

Thus,

$$e^{rt} = e^{\frac{-\beta \pm i\sqrt{4\rho_0\lambda T_0 - \beta^2}}{2\rho_0}}$$

General solution:

$$h(t) = e^{-\frac{\beta}{2\rho_0}t}\left(a_n \cos \frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0 - \beta^2}}{2\rho_0} t + b_n \sin \frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0 - \beta^2}}{2\rho_0} t\right)$$

Use Superposition to solve for initial conditions:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(a_n \cos \frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0 - \beta^2}}{2\rho_0} t + b_n \sin \frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0 - \beta^2}}{2\rho_0} t\right) \sin \frac{n\pi}{L} x$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x = f(x)$$

$\frac{\partial u}{\partial t}(x,0) = ?$ ← for next time

also look at question 4.4.9