4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0} t} \left(a_n \cos \frac{\sqrt{4\rho_0 (n^2\pi^2/L^2) T_0 - \beta^2}}{2\rho_0} t + b_n \sin \frac{\sqrt{4\rho_0 (n^2\pi^2/L^2) T_0 - \beta^2}}{2\rho_0} t \right) \sin \frac{n\pi}{L} x$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x = f(x)$$

Solve for the a_n's:

$$\sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin \frac{n\pi x}{L} x \sin \frac{m\pi x}{L} dx}_{\text{orthogonality of sin on } [0,L]} = \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$

by orthogonality the only term that survives in the sum is $m = n$.

$$a_m \frac{L}{2} = \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$

$$\boxed{a_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx}$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(a_n\cos\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}t + b_n\sin\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}t\right)\sin\frac{n\pi}{L}x$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\frac{-\beta}{2\rho_0}e^{-\frac{\beta}{2\rho_0}t}\left(a_n\cos\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}t + b_n\sin\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}t\right)\sin\frac{n\pi}{L}x$$

$$+\sum_{n=1}^{\infty}e^{-\frac{\beta}{2\rho_0}t}\left(-a_n\sin\underbrace{\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}}_{\omega_n}t + b_n\cos\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}t\right)\frac{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}{2\rho_0}\sin\frac{n\pi}{L}x$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\frac{-\beta}{2\rho_0}e^{-\frac{\beta}{2\rho_0}t}\left(a_n\cos\omega_n t + b_n\sin\omega_n t\right)\sin\frac{n\pi}{L}x$$

$$+\sum_{n=1}^{\infty}e^{-\frac{\beta}{2\rho_0}t}\left(-a_n\sin\omega_n t + b_n\cos\omega_n t\right)\omega_n\sin\frac{n\pi}{L}x$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\left(\frac{-\beta}{2\rho_0}a_n + b_n\omega_n\right)\sin\frac{n\pi}{L}x = g(x)$$

by orthogonality

$$\frac{-\beta}{2\rho_0}a_n + b_n\omega_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$b_n\omega_n = \frac{\beta}{2\rho_0}a_n + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{\omega_n}\left(\frac{\beta}{2\rho_0}a_n + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx\right)$$

$$\boxed{b_n = \frac{2\rho_0}{\sqrt{4\rho_0(n^2\pi^2/L^2)T_0-\beta^2}}\left(\frac{\beta}{2\rho_0}a_n + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx\right)}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

One more example:

4.4.9. From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L, \qquad (4.4.15)$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \qquad (4.4.1)$$

The energy $E = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$

Start by differentiating E by t and then simplify to try and get the right side

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)$$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \frac{1}{2}\int_0^L 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx + \frac{c^2}{2}\int_0^L 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx$$

$$= \int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx$$

$$= \int_0^L \left(\frac{\partial x}{\partial t}\, c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx$$

$$= c^2 \int_0^L \left(\frac{\partial x}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx$$

$\vdots$

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L,$$

what happens if I differentiate $\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}$?

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}$$

$$= c^2 \int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) dx = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L$$

done...

Chapter 5: **Sturm–Liouville Eigenvalue Problems**

Example: Heat equation:

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x)$$

+ Homogeneous boundary conditions...

Does separation of variables and superposition still work

$u(x,t) = \phi(x)\,h(t)$ plug it in...

$$c(x)\rho(x)\,\phi(x)h'(t) = \frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)h(t)\right) + Q(x,t)$$

$$c(x)\rho(x)\,\phi(x)h'(t) = h(t)\frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)\right) + Q(x,t)$$

what's this

can't just have any Q

$Q=0$ what we did before.

or more general.

$Q(x) = \alpha(x)\,u(x,t)$

$$c(x)\rho(x)\,\phi(x)\,h'(t) = h(t)\,\frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)\right) + \alpha(x)\,\phi(x)\,h(t)$$

divide by $h(t)$

$$\frac{c(x)\rho(x)\,\phi(x)\,h'(t)}{h(t)} = \frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)\right) + \alpha(x)\,\phi(x)$$

divide by $c(x)\rho(x)\phi(x)$

$$\frac{h'(t)}{h(t)} = \frac{\frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)\right) + \alpha(x)\,\phi(x)}{c(x)\rho(x)\,\phi(x)} = -\lambda$$

Get the ODEs:

$$h'(t) = -\lambda\, h(t) \qquad \frac{d}{dx}\left(K_0(x)\phi'(x)\right) + \alpha(x)\,\phi(x) = -\lambda\, c(x)\rho(x)\,\phi(x)$$

solve

$$h(t) = h(0)\,e^{-\lambda t}$$

in order that things don't exponentially increase in time intuitively the $\lambda > 0$.

What can we say about the solutions to this equation?

$$\underbrace{\frac{d}{dx}\left(K_0(x)\phi'(x)\right) + \alpha(x)\,\phi(x)} = -\lambda\, c(x)\rho(x)\,\phi(x)$$

self adjoint differential operation with respect to the boundary conditions used in the problem

In linear algebra A is self adjoint meant $A^T = A$.

$$Ax\cdot y = x\cdot A^T y = x\cdot Ay$$

$$L = \frac{d}{dx}K_0(x)\frac{d}{dx} + \alpha(x)$$

$$L\phi(x) = \left(\frac{d}{dx}K_0(x)\frac{d}{dx} + \alpha(x)\right)\phi(x) = \frac{d}{dx}\left(K_0(x)\phi'(x)\right) + \alpha(x)\,\phi(x)$$

means $u \cdot v$

Goal: Find an inner product (u,v) such that

$$(Lu, v) = (u, Lv)$$

where u and v are solutions to the ODE... actually only need u and v to satisfy the boundary conditions for the ODE...

5.3.1 General Classification

ODE.

$$\boxed{\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0,}$$

$p = p(x)$
$q = q(x)$
λ = eigenvalue
$\sigma = \sigma(x)$

recall non-uniform heat equation

$$\frac{d}{dx}\left(K_0(x)\phi'(x)\right) + \alpha(x)\phi(x) = -\lambda\, c(x)\rho(x)\,\phi(x)$$

$p(x) = K_0(x)$
$q(x) = \alpha(x)$
$\sigma(x) = c(x)\rho(x)$

BC.

$$\boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}}$$

Given this ODE and BC. then:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0,$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots.$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\,d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

Next time start thinking about these bullet points...