weight

eigenfunction

ODE: $$\underbrace{\frac{d}{dx}(p\phi')+q\phi}_{\text{differential operator}}+\underbrace{\lambda}_{\text{eigenvalue}}\sigma\phi=0 \qquad \text{for} \quad x\in[a,b]$$

BC: $\beta_1\phi(a)+\beta_2\phi'(a)=0$ homogeneous
$\beta_3\phi(b)+\beta_4\phi'(b)=0$

here β_i are constants and at least one is non zero per equation...

In order to talk about self adjoint I need a differential operator and some boundy conditions and a dot product.

$$Ly=\frac{d}{dx}(py')+qy$$ like the matrix A in linear alg.

Dot product

$$(u,v)=\int_a^b uv\,dx$$ like $x\cdot y$ in linear algebra

Self adjoint means

$$(Lu,v)=(u,Lv)$$ like $Ax\cdot y=x\cdot Ay$ exactly when $A^T=A$ in linear alg.

means $L^\dagger=L$ or $L^*=L$

need to explain why this is true for the differential operator

$$Ly=\frac{d}{dx}(py')+qy$$

when u and v satisfy the boundary conditions

$\beta_1u(a)+\beta_2u'(a)=0$ $\qquad$ $\beta_1v(a)+\beta_2v'(a)=0$
$\beta_3u(b)+\beta_4u'(b)=0$ $\qquad$ $\beta_3v(b)+\beta_4v'(b)=0$

Need to show

$$(Lu,v)-(u,Lv)=0$$

$$(Lu,v)-(u,Lv)=\int_a^b (Lu)v\,dx-\int_a^b uLv\,dx$$
$$=\int_a^b (vLu-uLv)\,dx$$

$$vLu = v\left(\frac{d}{dx}(pu')+qu\right)=v\frac{d}{dx}(pu')+\cancel{vqu}$$
$$uLv = u\left(\frac{d}{dx}(pv')+qv\right)=u\frac{d}{dx}(pv')+\cancel{uqv}$$

sub-tract

move it out

$$(Lu,v)-(u,Lv)=\int_a^b\left(v\frac{d}{dx}(pu')-u\frac{d}{dx}(pv')\right)dx$$

$$v\frac{d}{dx}(pu')=\frac{d}{dx}(vpu')-\cancel{v'pu'}$$
sub-tract
$$u\frac{d}{dx}(pv')=\frac{d}{dx}(upv')-\cancel{u'pv'}$$

$$(Lu,v)-(u,Lv)=\int_a^b\left(\frac{d}{dx}(vpu')-\frac{d}{dx}(upv')\right)dx$$

$$=\cancel{\int_a^b}\cancel{\frac{d}{dx}}\left(p\left(vu'-uv'\right)\right)dx = p(vu'-uv')\Big|_a^b = 0-0=0$$

$u'(a)=-\frac{\beta_1}{\beta_2}u(a)$ $\quad v'(a)=-\frac{\beta_1}{\beta_2}v(a)$

the general boundary conditions need to be designed so that this term is zero...

BC: $\beta_1 u(a)+\beta_2 u'(a)=0$ $\quad \beta_1 v(a)+\beta_2 v'(a)=0$

$\beta_3 u(b)+\beta_4 u'(b)=0$ $\quad \beta_3 v(b)+\beta_4 v'(b)=0$

assume all the β_i's are non-zero (though a similar argument works when only one β_i is not zero per condition)

$$p(a)\cdot(v(a)u'(a)-u(a)v'(a))=p(a)\left(v(a)\left(\cancel{-\frac{\beta_1}{\beta_2}}u(a)\right)-u(a)\left(\cancel{-\frac{\beta_1}{\beta_2}}v(a)\right)\right)=0$$

$$p(b)\cdot\left(v(b)u'(b) - u(b)v'(b)\right) = p(b)\left(v(b)\left(-\frac{\beta_3}{\beta_4}u(b)\right) - u(b)\left(-\frac{\beta_3}{\beta_4}v(b)\right)\right) = 0$$

Therefore,

$$(Lu, v) - (u, Lv) = 0$$

so $\quad (Lu, v) = (u, Lv)\quad$ which means L is self adjoint.

Consequences...

> 5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
> $$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
> 6. Any eigenvalue can be related to its eigenfunction by the

this is a different inner product than we just used to show L was self adjoint...

$$(u, v)_\sigma = \int_a^b u v \sigma \, dx$$

Eigenfunctions mean a solution to this differential equation

$$\boxed{\frac{d}{dx}(p\phi') + q\phi} + \lambda\sigma\phi = 0$$

and these boundary conditions

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0 \quad 2$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0$$

In terms of L we have

$$L\phi + \lambda\sigma\phi = 0 \quad \text{or} \quad L\phi = -\lambda\sigma\phi.$$

Let λ_n be the eigenvalue for ϕ_n
and λ_m ———— " ———— ϕ_m.

Thus $\quad L\phi_n = -\lambda_n\sigma\phi_n \quad$ and $\quad L\phi_m = -\lambda_m\sigma\phi_m.$

By the self adjointedness ~~of~~ λ_n ~~of~~ ϕ_n we know

$$0 = \int_a^b (\phi_m L \phi_n - \phi_n L \phi_m)\, dx$$

since the eigenfunctions satisfy the boundary conditions.

Substituting yields

$$\int_a^b \left(\phi_m (-\lambda_n \sigma \phi_n) - \phi_n (-\lambda_m \sigma \phi_m)\right) dx$$

$$= \int_a^b (-\lambda_n \phi_m \sigma \phi_n + \lambda_m \phi_n \sigma \phi_m)\, dx$$

$$= (\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma \, dx = 0$$

since $\lambda_m \neq \lambda_n$ by assumption then

$$\int_a^b \phi_m \phi_n \sigma \, dx = 0$$

weight function appears and different the the similar result from linear algebra...

the orthogonality relationship

1. All the eigenvalues λ are real.

For contradiction

Suppose $L\phi + \lambda \sigma \phi = 0$ for $\boxed{\text{some } \lambda \text{ that was not real}}$ contradicted this

If λ has a non-zero imaginary part tha $\bar{\lambda} \neq \lambda$

$$\overline{L\phi + \lambda \sigma \phi} = \bar{0}$$

$$\bar{L}\bar{\phi} + \bar{\lambda}\bar{\sigma}\bar{\phi} = \bar{0}$$

$$L\bar{\phi} + \bar{\lambda}\sigma\bar{\phi} = 0$$

this means tha $\bar{\phi}$ is an eigenfunction with eigenvalue $\bar{\lambda}$.

Since ϕ and $\bar{\phi}$ have different eigenvalues they are othogonal

$$0 = (\varphi, \bar{\varphi})_\sigma = \int_a^b \varphi \bar{\varphi} \sigma \, dx = \int_a^b |\varphi|^2 \sigma \, dx$$

since φ is an eigenvector it's ~~not~~ identically zero therefore

$$0 = \int_a^b |\varphi|^2 \sigma \, dx > 0$$

which is a contradiction...

Therefore λ is real. That all eigenvalues are real for a self-adjoint operator..

3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any

By contradiction suppose for a single eigenvalue λ there were two different eigenfunctions: φ_1 and φ_2.

$$L\varphi_1 + \lambda\sigma\varphi_1 = 0 \quad \text{and} \quad L\varphi_2 + \lambda\sigma\varphi_2 = 0$$

$$\varphi_2 L\varphi_1 - \varphi_1 L\varphi_2 = \frac{d}{dx}\left(p\left(\varphi_2\varphi_1' - \varphi_1\varphi_2'\right)\right)$$

$$\parallel$$

$$\varphi_2(-\cancel{\lambda\sigma\varphi_1}) - \varphi_1(-\cancel{\lambda\sigma\varphi_2}) = 0$$

Thus

$$\frac{d}{dx}\left(p\left(\varphi_2\varphi_1' - \varphi_1\varphi_2'\right)\right) = 0$$

means

$$p\left(\varphi_2\varphi_1' - \varphi_1\varphi_2'\right) = \text{const.}$$

and the const. is the same as when $x = a$ at the boundary. and the boundary conditions were designed so this is 0.

$$p(a)\left(\phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a)\right) = 0$$

So const. $= 0$ and

$$p\left(\phi_2\phi_1' - \phi_1\phi_2'\right) = 0 \quad \text{everywhere...}$$

$$\boxed{\phi_2\phi_1' - \phi_1\phi_2' = 0} \quad \text{everywhere...}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{\phi_1}{\phi_2}\right) = \frac{\boxed{\phi_1'\phi_2 - \phi_1\phi_2'}}{\phi_2^2} = 0$$

This means $\frac{\phi_1}{\phi_2} = c$ where c is a constant.

Therefore $\phi_1 = c\,\phi_2$ and we see that ϕ_1 and ϕ_2 where really the same eigenfunction, just rescaled.

(this finishes #3.)