also true for symmetric matrices

1. All the eigenvalues λ are real.

skip...

2. There exist an infinite number of eigenvalues:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots .$$

   a. There is a smallest eigenvalue, usually denoted λ_1.

   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.

3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

difficult...

4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).

also true for symmetric matrices. means there is an orthogonal basis of eigenvectors or functions...

5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\ dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

spectral theorem for self adjoint operators (or matrices).

easy...

6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:

$$\lambda = \frac{-p\phi\ d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\ dx}{\int_a^b \phi^2 \sigma\ dx},$$

   where the boundary conditions may somewhat simplify this expression.

also defined for matrices...

(6) Consider matrix eigenvalue-eigenvector problem:

Math 330: $Ax = \lambda x$

used determinants to eliminate x from the equation:

$$\det(A - \lambda I) = 0$$

solve for λ. Now that you know λ plug it back in an solve for x

$x \in \mathrm{Null}(A - \lambda I) \setminus \{0\}$.

where λ is the eigenvalue.

What if we somehow knew x. How to find λ:

- Plug λ in and solve for x.

- It does happen in numerical linear algebra that some algorithm finds the x first.. Then we need to find λ.
- From observations, we might just see an eigenfunction directly...

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

for example as $t \to \infty$ the solution looks like the leading term

$$u(x,t) \sim A_1 \underbrace{\cos\frac{\pi x}{L}}_{\text{eigenfunction...}} e^{-k\left(\frac{\pi}{L}\right)^2 t}$$

linear algebra:

$$\boxed{Ax = \lambda x}$$

view as least squares optimization

solve for λ

idea ... use the inner product

$$Ax \cdot x = \lambda \underbrace{x \cdot x}$$

is just a number, $x \cdot x > 0$ since by definition $x \neq 0$.

$$\lambda = \frac{Ax \cdot x}{x \cdot x}$$

solve for λ.

$Ax = \lambda x$ } there are n equations, $x \in \mathbb{R}^n$

λ unknown

find λ that minimizes $\| Ax - \lambda x \|$.

rewrite so it looks like a least squares problem

$$Ax = \lambda x$$

$$x\lambda = \underbrace{Ax}$$

$A \quad x \approx b$ ← find x to minimize $Ax = b$
solve $\underbrace{A^T A x = A^T b}$
normal equations...

blue variables are different that the black variables

$$A^T A x = A^T b$$

$$x^T x \lambda = x^T A x$$

$$x \cdot x \, \lambda = x \cdot Ax = Ax \cdot x$$

$$\lambda = \frac{Ax \cdot x}{x \cdot x}$$ least squares solution for λ.

Note the least squares solution is the exact solution if there actually is a solution..

This means $\min\{\|Ax-\lambda x\| : \lambda \in \mathbb{R}\} = 0$

So λ is the exact solution and $Ax = \lambda x$...

eigenvalue...

Now translate all of this to differential operators and eigenfunctions...

$$L(\phi) + \lambda\sigma(x)\phi = 0,$$

Not only L is self adjoint but it's a Sturm-Liouville operator...

$$L(y) \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y.$$

Given φ solve for λ. Take inner products

$$(u,v) = \int_a^b u(x)v(x)\,dx \quad \text{means...} = u \cdot v$$

$$(L(\varphi),\varphi) + (\lambda\sigma\varphi,\varphi) = 0$$

just a number....

$$(L(\varphi),\varphi) + \lambda(\sigma\varphi,\varphi) = 0$$

solve for λ

$$\lambda = \frac{-(L(\varphi),\varphi)}{(\sigma\varphi,\varphi)}$$

Raleigh Quotient for λ

$$L(y) \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y.$$

Now simplify this

$$-(L(\phi),\phi) = -\left(\frac{d}{dx}(p\phi') + q\phi, \phi\right)$$

$$= -\int_a^b \left(\frac{d}{dx}(p\phi') + q\phi\right)\phi \, dx$$

$$= -\int_a^b \left[\frac{d}{dx}(p\phi')\right]\phi \, dx \boxed{- \int_a^b q\phi^2 dx}$$

$$-\int_a^b \left[\frac{d}{dx}(p\phi')\right]\phi \, dx = -\left(uv\Big|_a^b - \int_a^b v\, du\right)$$

$u = \phi$ $\qquad du = \phi' dx$

$dv = \left[\frac{d}{dx}(p\phi')\right] dx$ $\qquad v = \int dv = \int\left[\frac{d}{dx}(p\phi')\right] dx = p\phi'$

fund theorem of Calculus

$$-\int_a^b \left[\frac{d}{dx}(p\phi')\right]\phi \, dx = -\phi p\phi'\Big|_a^b + \int_a^b p(\phi')^2 dx$$

Therefore

$$\lambda = \frac{-(L(\phi),\phi)}{(\sigma\phi,\phi)} \approx \frac{-\phi p\phi'\Big|_a^b + \int_a^b p(\phi')^2 dx - \int_a^b q\phi^2 dx}{\int_a^b \sigma\phi^2 dx}$$

...quotient: zero by boundary conditions... might be negative... unless $q<0$ then it's positive...

$$\lambda = \frac{-p\phi\, d\phi/dx|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2\sigma\, dx},$$

positive

The reason integration by parts is usefull is because

$$\int_a^b p(x)(\phi'(x))^2\,dx \geq 0 \quad \text{since } (\phi')^2 \geq 0$$

and in many application $p > 0$,

For example

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x)$$

plays the role of $p(x)$ and this is positive

There's a homework problem where you might want to use the Raleigh quotient to determine the sign of λ.

This one...

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

Next time...

5.4 WORKED EXAMPLE: HEAT FLOW IN A NONUNIFORM ROD WITHOUT SOURCES