$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

$a < x < b,$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

$p > 0$ and $\sigma > 0$ everywhere

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots.$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\ dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\ d\phi/dx|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\ dx}{\int_a^b \phi^2\sigma\ dx},$$
where the boundary conditions may somewhat simplify this expression.

PDE: $$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)}$$

$c = c(x)$
$\rho = \rho(x)$
$K_0 = K_0(x)$

BC: $$\boxed{\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \end{aligned}}$$

Idea: separation of variables
superposition
orthogonality

IC: $$\boxed{u(x,0) = f(x).}$$

Let $u(x,t) = \phi(x)h(t)$
and plug it in...

$$c(x)\rho(x)\,\phi(x)\,h'(t) = \frac{\partial}{\partial x}\left(K_0(x)\,\phi'(x)\,h(t)\right)$$

$$c(x)\rho(x)\,\phi(x)\,h'(t) = h(t)\,\frac{d}{dx}\left(K_0(x)\,\phi'(x)\right)$$

$$\frac{h'(t)}{h(t)} = \frac{\frac{d}{dx}\left(K_0(x)\phi'(x)\right)}{c(x)\rho(x)\phi(x)} = -\lambda$$

Get two ODEs...

$$h'(t) = -\lambda h(t)$$

solve

$$h(t) = c\,e^{-\lambda t}$$

a_n

$$\frac{d}{dx}\left(K_0(x)\phi'(x)\right) + \lambda\, c(x)\rho(x)\phi(x) = 0$$

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \cancel{q(x)\phi} + \lambda\sigma(x)\phi = 0,$$

note also $p>0$ and $q\geq 0$

apply theorem about Sturm-Liouville problems here...

$$u(0,t) = 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0$$

$$\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0$$
$$\beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0,$$

Check boundary conditions $a=0$, $b=L$
$\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 0$, β_4
shows they are of the form needed by the Theorem.

by (2) and (3) there are an infinite # of eigenvalues λ_n with corresponding eigenfunctions ϕ_n such that

$$\frac{d}{dx}\left(K_0(x)\phi_n'(x)\right) + \lambda_n c(x)\rho(x)\phi_n(x) = 0$$

That is this ODE has lots of solutions...

Superposition principle
$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

Note that $u(x,t)$ satisfies the boundary conditions because they were homogeneous and because each of the ϕ_n's satisfy them.

Now I want to solve for the initial condition

$$u(x,0) = f(x).$$

Thus

$$u(x,0) = \sum_{n=1}^{\infty} a_n \varphi_n(x) = f(x)$$

Use orthogonality (5) to solve for the a_n's under the assumption that $f(x)$ can even be expressed this way

(4) piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Furthermore this infinite series converges to $[f(x+) + f(x-)]/2$

To (4) in the theorem say $f(x)$ can be written this way provided its piecewise smooth.

weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\, dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

Thus

$$\int_0^L \varphi_n(x)\, \underbrace{\varphi_m(x)\, c(x)\rho(x)}\, dx = 0 \qquad \text{if } \lambda_n \neq \lambda_m$$

Use orthogonality to solve for the a_n's so that

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) = f(x)$$

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)\varphi_m(x)\, c(x)\rho(x) = f(x)\varphi_m(x)\, c(x)\rho(x)$$

$$\int_0^L \sum_{n=1}^{\infty} a_n \varphi_n(x)\varphi_m(x)\, c(x)\rho(x)\, dx = \int_0^L f(x)\varphi_m(x)\, c(x)\rho(x)\, dx$$

$$\sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \varphi_n(x)\,\varphi_m(x)\,c(x)\rho(x)\,dx}_{\text{non zero only when } n=m.} = \int_0^L f(x)\,\varphi_m(x)\,c(x)\rho(x)\,dx$$

If $n=m$

$$\int_0^L \varphi_n(x)\,\varphi_m(x)\,c(x)\rho(x)\,dx = \int_0^L \varphi_m(x)^2\,c(x)\rho(x)\,dx$$

$$a_m \int_0^L \varphi_m(x)^2\,c(x)\rho(x)\,dx = \int_0^L f(x)\,\varphi_m(x)\,c(x)\rho(x)\,dx$$

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n\,\varphi_n(x)\,e^{-\lambda_n t}$$

where

$$a_m = \frac{\int_0^L f(x)\,\varphi_m(x)\,c(x)\rho(x)\,dx}{\int_0^L \varphi_m(x)^2\,c(x)\rho(x)\,dx}$$

What about ① and ⑥ in the theorem? Useful for interpretation... λ_n are real.

Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:

$$\lambda_n = \frac{-p\phi_n\,d\phi_n/dx\big|_0^L + \int_0^L [p(d\phi_n/dx)^2 - q\phi_n^2]\,dx}{\int_0^L \phi_n^2\sigma\,dx},$$

where the boundary conditions may somewhat simplify this expression.

Boundary conditions

$$\boxed{\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \end{aligned}}$$ $\leftarrow \phi_n(0) = 0$ $\leftarrow \phi_n'(L) = 0$

$$-K_0 \phi_n \phi_n' \Big|_0^L = -K_0(L)\phi_n(L)\phi_n'(L) + K_0(0)\phi_n(0)\phi_n'(0) = 0$$

$$\lambda_n = \frac{\int_0^L K_0(x)\,\phi_n'(x)^2\,dx}{\int_0^L \phi_n(x)^2\,c(x)\,\rho(x)\,dx}$$

Since $\phi_n(x) = 0$ is never an eigenfunction then

$$\int_0^L \phi_n(x)^2\,c(x)\,\rho(x)\,dx > 0$$

What if $\phi_n'(x) = 0$? Could it happen?

If it did that means $\phi_n(x) = \text{const.}$

$\phi_n(0) = 0 \leftarrow$ Boundary cond...

Thus $\phi_n(x) = 0$ ... which is not eigenfunction ...

Conclusion $\phi_n'(x) \neq 0$.

Thus

$$\int_0^L K_0(x)\,\phi_n'(x)^2\,dx > 0$$

positive square of non-zero

Therefore

$$\lambda_n = \frac{\int_0^L K_0(x)\,\phi_n'(x)^2\,dx}{\int_0^L \phi_n(x)^2\,c(x)\,\rho(x)\,dx} > 0$$

Thus every term in the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

exponentially decays to zero

Qualitatively the behavior as $t \to \infty$ is similar to the constant coefficient heat equation with same boundary conditions...

More stuff about Raleigh quotient...

Linear algebra case... Spectral theorem for symmetric Matrices...

If $A = A^T$ then there is an orthonormal basis of eigenvectors x_n with real eigenvalues λ_n such that

$$A x_n = \lambda_n x_n \quad \text{and} \quad x_n \cdot x_m = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Recall Raleigh Quotient

$$Ax \cdot x = \lambda x \cdot x$$

$$R(x) = \frac{Ax \cdot x}{x \cdot x}$$

Note: if $x = x_n$ then $R(x_n) = \lambda_n$

If $x \neq x_n$ then since they form a basis

$$x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$R(x) = \frac{A(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) \cdot (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)}{(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) \cdot (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)}$$

$$R(x) = \frac{(c_1\lambda_1 x_1 + c_2\lambda_2 x_2 + \cdots + c_n\lambda_n x_n)\cdot(c_1x_1 + c_2x_2 + \cdots + c_nx_n)}{(c_1x_1 + c_2x_2 + \cdots + c_nx_n)\cdot(c_1x_1 + c_2x_2 + \cdots + c_nx_n)}$$

$$= \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2}{c_1^2 + c_2^2 + \cdots + c_n^2}$$

this is a weighted average of the λ_i's.

$$\min\{\lambda_i\} \leq \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2}{c_1^2 + c_2^2 + \cdots + c_n^2} \leq \max\{\lambda_i\}$$

Order the eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$$

then

$$\lambda_1 = \min\{R(x) : x \in \mathbb{R}^n\}$$

$$\lambda_n = \max\{R(x) : x \in \mathbb{R}^n\}.$$

For Sturm-Liouville problems the same works, except not for the max eigenvalue because they go to ∞.

$$\lambda_1 \leq RQ[u_T] = \frac{-pu_T\, du_T/dx\big|_a^b + \int_a^b \left[p\,(du_T/dx)^2 - qu_T^2\right]\, dx}{\int_a^b u_T^2\sigma\, dx}. \qquad (5.6.6)$$

smallest eigenvalue..

bounded above by the Raleigh Quotient of any function that satisfies the boundary condition.

One gets exact equality by minimizing...

$$\boxed{\lambda_1 = \min \frac{-pu\, du/dx\big|_a^b + \int_a^b \left[p\,(du/dx)^2 - qu^2\right]\, dx}{\int_a^b u^2\sigma\, dx},}$$

Back to PDE: knowing λ_1 tells how fast

$$u(x,t) = \sum_{n=1}^{\infty} a_n\, \phi_n(x)\, e^{-\lambda_n t}$$

exponentially decays to zero

decays because λ_1 is the slowest decaying term in the series.