20- Apr - 2023

## [~~16-Feb-2023~~] Quiz 2

april 20

Quiz 2 will be given in class on ~~February 16~~. Please be prepared to solve the Laplace equation as in Exercise 2.5.1 from the text and the wave equation as in Exercises 4.4.2 and 4.4.3.

**2.5.1.** Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in $y$, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in $x$, let $u(x,y) = \phi(x)h(y)$.]:

*(a) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = f(x)$

(b) $\frac{\partial u}{\partial x}(0,y) = g(y)$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$

*(c) $\frac{\partial u}{\partial x}(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

(d) $u(0,y) = g(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = 0$

*(e) $u(0,y) = 0$, $u(L,y) = 0$, $u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0$, $u(x,H) = f(x)$

(f) $u(0,y) = f(y)$, $u(L,y) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$

(g) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$, $\frac{\partial u}{\partial y}(x,H) = 0$

(h) $u(0,y) = 0$, $u(L,y) = g(y)$, $u(x,0) = 0$, $u(x,H) = 0$

**4.4.2.** In Section 4.2 it was shown that the displacement $u$ of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where $Q$ represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

**(a)** Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

**(b)** Separate variables if $\rho_0(x)$ and $\alpha(x)$ but $T_0$ is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

***(c)** Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0,t) = 0, \qquad u(x,0) = 0$$

$$u(L,t) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

**4.4.3.** Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

**(a)** Briefly explain why $\beta > 0$.

***(b)** Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient $\beta$ is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

not homogeneous

*(e) $u(0,y)=0,\quad u(L,y)=0,\quad u(x,0)-\frac{\partial u}{\partial y}(x,0)=0,\quad u(x,H)=f(x)$

$$u(x,y)=\varphi(x)h(y)$$

Laplace equation $\quad u_{xx}+u_{yy}=0 \quad$ or $\quad \Delta u=0$

$$\varphi''(x)h(y)+\varphi(x)h''(y)=0$$

$$\frac{\varphi''(x)}{\varphi(x)}=-\frac{h''(y)}{h(y)}=-\lambda$$

Thus,

$$\varphi''(x)=-\lambda\varphi(x) \quad \text{and} \quad h''(y)=\lambda h(y)$$

$$\varphi(0)=0,\quad \varphi(L)=0, \qquad h(0)-h'(0)=0,\quad \boxed{h(H)=c}$$

Solve for $\varphi$:

$$\varphi(x)=a\cos\sqrt{\lambda}\,x+b\sin\sqrt{\lambda}\,x$$

$\varphi(0)=a=0 \quad$ so $a=0$

$\varphi(L)=b\sin\sqrt{\lambda}L=0$

Since $b\neq 0$ then $\sqrt{\lambda}L=n\pi$

$$\sqrt{\lambda}=\frac{n\pi}{L}$$

$$\lambda=\left(\frac{n\pi}{L}\right)^2$$

Next solve for $h$...

$$h(y)=\alpha e^{\sqrt{\lambda}y}+\beta e^{-\sqrt{\lambda}y}$$

General idea ... substitute $e^{rt}$ to obtain

$$r^2e^{rt}=\lambda e^{rt}$$

$r^2=\lambda$ so $r=\pm\sqrt{\lambda}$

relationship for $\alpha$ and $\beta$ so that

$$h(0)-h'(0)=0$$

$$h'(y)=\alpha\sqrt{\lambda}e^{\sqrt{\lambda}y}-\beta\sqrt{\lambda}e^{-\sqrt{\lambda}y}$$

$$h(0)-h'(0)=\alpha+\beta-\alpha\sqrt{\lambda}+\beta\sqrt{\lambda}=0$$

$$\alpha(1+\sqrt{\lambda})-\beta(1-\sqrt{\lambda})=0 \qquad \alpha=\beta\left(\frac{1-\sqrt{\lambda}}{\sqrt{\lambda}+1}\right)$$

Therefore by superposition

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n \sin\frac{n\pi x}{L}\left[\left(\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)e^{\frac{n\pi}{L}y} + e^{-\frac{n\pi}{L}y}\right]$$

Panic... should have use hyperbolic functions, maybe...

Quick check if better... $\boxed{\text{yes}}$ worth checking... ☺

Next solve for h...

$$h(y) = \alpha e^{\sqrt{\lambda}y} + \beta e^{-\sqrt{\lambda}y}$$

$$h(y) = \gamma\cosh\sqrt{\lambda}y + \delta\sinh\sqrt{\lambda}y$$

$$h'(y) = \sqrt{\lambda}\gamma\sinh\sqrt{\lambda}y + \sqrt{\lambda}\delta\cosh\sqrt{\lambda}y$$

$$h(0) - h'(0) = \gamma - \sqrt{\lambda}\delta = 0 \qquad \gamma = \sqrt{\lambda}\delta$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

Therefore by superposition

$$u(x,t) = \sum_{n=1}^{\infty} \delta_n \sin\frac{n\pi x}{L}\left[\sqrt{\lambda}\cosh\frac{n\pi y}{L} + \sinh\frac{n\pi y}{L}\right]$$

$$\cancel{= \sum_{n=1}^{\infty} \delta_n \sin\frac{n\pi x}{L} e^{\frac{n\pi y}{L}}}$$

Find the coefficient $\delta_n$ to satisfy the remaining boundary condition

$$u(x,H) = f(x)$$

$$u(x,H) = \sum_{n=1}^{\infty} \delta_n \sin\frac{n\pi x}{L}\underbrace{\left[\frac{n\pi}{L}\cosh\frac{n\pi H}{L} + \sinh\frac{n\pi H}{L}\right]}_{\text{constant}} = f(x)$$

$$\int_0^L \sum_{n=1}^{\infty} \delta_n \sin\frac{n\pi x}{L}\left[\frac{n\pi}{L}\cosh\frac{n\pi H}{L} + \sinh\frac{n\pi H}{L}\right]\sin\frac{m\pi x}{L}\,dx = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

$$\frac{L}{2}\delta_m\left[\frac{m\pi}{L}\cosh\frac{m\pi H}{L} + \sinh\frac{m\pi H}{L}\right] = \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$$

$$\delta_n = \frac{2}{L}\,\frac{1}{\left[\frac{n\pi}{L}\cosh\frac{n\pi H}{L} + \sinh\frac{n\pi H}{L}\right]}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty} \delta_n \sin\frac{n\pi x}{L}\left[\sqrt{\lambda}\cosh\frac{n\pi y}{L} + \sinh\frac{n\pi y}{L}\right]$$

$\infty$

where

$$\delta_n = \frac{2}{L}\,\frac{1}{\left[\frac{n\pi}{L}\cosh\frac{n\pi H}{L} + \sinh\frac{n\pi H}{L}\right]}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

CHAPTER 12

# The Method of Characteristics for Linear and Quasilinear Wave Equations

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.}$$

s that it can be "factored" in two ways:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0,$$

erivative terms vanish in both. If we let

$$\boxed{w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}}$$

$$\boxed{v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x},}$$

$$\begin{cases} \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) w = 0 \\ \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \end{cases}$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) w = 0 \qquad \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

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How to solve $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ for the function $w(x,t)$?

parameterize $x$ as a function of $t$
or $t$ ~~as a function of~~ $x$ to get an ODE

In this $x = x(t)$ and consider $w(x(t), t)$ which is now only a function of $t$.

$$\frac{d}{dt} w(x(t),t) = w_x(x(t),t)\, x'(t) + w_t(x(t),t)$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$$

If $x'(t) = c$ then $\frac{d}{dt} w(x(t),t) = 0$

First solve the ODE... to get the characterist directions

Then solve this ode along those directions

$x(t) = ct + x(0)$ plug in

$$\frac{d}{dt} w(ct + x(0), t) = 0$$

Thus $w(ct + x(0), t) = \text{const} = w(x(0), 0)$ at $t=0$

Solving this... but what was the initial condition

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0, \qquad w(x,0) = f(x)$$

Thus $w(\underbrace{ct + x}_{\alpha}, t) = f(x)$

This is the initial cond in terms of the solution
Want solution in terms of initial condition

if $\alpha = ct + x$
then $x = \alpha - ct$

$$\boxed{w(\alpha, t) = f(\alpha - ct)}$$

Solution in terms of the initial condition...

ordinary d.e.

$$\underbrace{\boxed{\frac{d}{dt} w(x(t),t)} = w_x(x(t),t)\, x'(t) + w_t(x(t),t) = 0}$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$$

Suppose $t$ as a funct of $x$

$w(x, t(x))$ is a function of $x$

$$\frac{d}{dx} w(x,t(x)) = \frac{\partial w}{\partial x}\frac{dx}{dx} + \frac{\partial w}{\partial t}\frac{dt}{dx} = \boxed{w_x + w_t\, t'(x)}$$

$$t'(x) = \frac{1}{c}$$