$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u}_{w} = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u}_{v} = 0$$

Let

$$w = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} \qquad v = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$$

the result is a system of first order PDEs

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)w = 0 \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = 0$$

or

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial t} = 0$$

solved this for w last time

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

for review solve this one

Method of characteristics to reduce the first order PDE into a system of 1st order ODEs

Suppose $x = x(t)$ and consider $V(x(t), t)$

this assumption may not always work... maybe try $t = t(x)$ otherwise...

Note $V(x(t),t)$ is only a function of t.

$$\frac{d}{dt} V(x(t),t) = \frac{\partial V}{\partial x}(x(t),t)\, x'(t) + \frac{\partial V}{\partial t}(x(t),t)$$

Compar to the PDE

$$\boxed{\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0}$$

Let $x'(t) = -c$ then $x(t) = -ct + x_0$

$$\frac{dV(x(t),t)}{dt} = \boxed{\frac{\partial V}{\partial x}(x(t),t)(-c) + \frac{\partial V}{\partial t}(x(t),t)} = 0$$

provided v is a solution to the PDE

Two ODE's.

$$\begin{cases} x'(t) = -c \\ \dfrac{d}{dt} v(x(t),t) = 0 \end{cases}$$

solution

$$\begin{cases} x(t) = -ct + x_0 \\ v(x(t),t) = \text{const.} \end{cases}$$

Thus

$$v(-ct+x_0, t) = \text{const}$$

put $t=0$ to find the constant from the initial condition $v(x,0) = g(x)$.

Thus,

$$v(x_0, 0) = g(x_0)$$

and

$$v(\underbrace{-ct + x_0}_{x}, t) = g(x_0)$$

rewrite this setting $x = -ct + x_0$

to obtain $x_0 = x + ct$

$$\boxed{v(x,t) = g(x+ct)}$$

last time

$$\boxed{w(x,t) = f(x-ct)}$$

Use these to solve the original PDE, i.e., the wave equation... Thus...

$$w = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} \qquad v = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$$

Plug in to get

$$\begin{cases} f(x-ct) = \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} \\ g(x+ct) = \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} \end{cases}$$

Simplify

$$\begin{cases} f(x-ct) + g(x+ct) = 2\dfrac{\partial u}{\partial t} \\ f(x-ct) - g(x+ct) = 2c\dfrac{\partial u}{\partial x} \end{cases}$$

$$2\left(u(x,t)-u(x,0)\right)=\int_0^t 2\frac{\partial u}{\partial t}(x,s)\,ds$$

$$2c\left(u(x,0)-u(0,0)\right)=\int_0^x 2c\frac{\partial u}{\partial x}(y,0)\,dy$$

Thus

$$2\left(u(x,t)-u(x,0)\right)=\int_0^t \left(f(x-cs)+g(x+cs)\right)ds$$

$$2c\left(u(x,0)-u(0,0)\right)=\int_0^x \left(f(y-c0)-g(y+c0)\right)dy$$

$$=\int_0^x f(y)\,dy-\int_0^x g(y)\,dy$$

and

$$2\left(u(x,t)-u(x,0)\right)=\int_0^t \left(f(x-cs)+g(x+cs)\right)ds$$

$$=\int_0^t f(x-cs)\,ds+\int_0^t g(x+cs)\,ds$$

$$\int_0^t f(x-cs)\,ds=-\frac{1}{c}\int_x^{x-ct} f(z)\,dz$$

$$z=x-cs$$

$$dz=-c\,ds$$

$$\int_0^t g(x+cs)\,ds = \frac{1}{c}\int_x^{x+ct} g(z)\,dz$$

$z = x+cs$

$dz = c\,ds$

$$\boxed{\begin{aligned} &2\left(u(x,t) - u(x,0)\right) = -\frac{1}{c}\int_x^{x-ct} f(z)\,dz + \frac{1}{c}\int_x^{x+ct} g(z)\,dz \\ &2c\left(u(x,0) - u(0,0)\right) = \int_0^x f(y)\,dy - \int_0^x g(y)\,dy \end{aligned}}$$

Solve for $u(x,t)$

$$2c\left(u(x,t) - \cancel{u(x,0)}\right) = -\int_x^{x-ct} f(z)\,dz + \int_x^{x+ct} g(z)\,dz$$

$$2c\left(\cancel{u(x,0)} - u(0,0)\right) = \int_0^x f(y)\,dy - \int_0^x g(y)\,dy$$

$$2c\left(u(x,t) - u(0,0)\right) = -\int_0^{x-ct} f(z)\,dz + \int_0^{x+ct} g(z)\,dz$$

$$u(x,t) = \underbrace{u(0,0)}_{\text{const.}} - \frac{1}{2c}\left[\underbrace{\int_0^{x-ct} f(z)\,dz}_{\text{function of } x-ct} + \underbrace{\int_0^{x+ct} g(z)\,dz}_{\text{function of } x+ct}\right]$$

In the end

$$u(x,t) = F(x-ct) + G(x+ct)$$

for example

$$F(x-ct) = u(0,0) - \frac{1}{2c}\int_0^{x-ct} f(z)\,dz$$

$$G(x+ct) = \frac{1}{2c}\int_0^{x+ct} g(z)\,dz$$

or if you like symmetry

$$F(x-ct) = \frac{1}{2}u(0,0) - \frac{1}{2c}\int_0^{x-ct} f(z)\,dz$$

$$G(x+ct) = \frac{1}{2}u(0,0) + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz$$

(forget these f's and g's.)

Solution to

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0.}$$

for $x \in \mathbb{R}$ and $t \geq 0$

can be written

$$u(x,t) = F(x-ct) + G(x+ct)$$

General solution where F and G are chosen to satisfy the initial conditions...

$$\boxed{u(x,t) = F(x-ct) + G(x+ct),}$$

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also $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(F(x-ct)+G(x+ct)\right)$

$= -cF'(x-ct)$

Initial conditions...

$$\boxed{u(x,0) = f(x),} \qquad -\infty < x < \infty$$

note these f's and g's are different than the ones before

$$\boxed{\frac{\partial u}{\partial t}(x,0) = g(x),} \qquad -\infty < x < \infty.$$

$$u(x,0) = F(x) + G(x) = f(x)$$

also $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(F(x-ct)+G(x+ct)\right)$

$$\frac{\partial u}{\partial t} = -cF'(x-ct) + cG'(x+ct)$$

Solve these for F and G

and $\frac{\partial u}{\partial t}(x,0) = -cF'(x) + cG'(x) = g(x)$

$$\boxed{\begin{aligned} F'(x) + G'(x) &= f'(x) \\ -F'(x) + G'(x) &= \frac{g(x)}{c} \end{aligned}}$$

So

$$2F'(x) = f'(x) - \frac{g(x)}{c}$$

$$2G'(x) = f'(x) + \frac{g(x)}{c}$$

Integrate to get $F(x)$ and $G(x)$

$$2(F(x)-F(0)) = \int_0^x 2F'(y)\,dy$$

$$= \int_0^x \left(f'(y) - \frac{g(y)}{c}\right) dy$$

$$= f(x) - f(0) - \frac{1}{c}\int_0^x g(y)\,dy$$

$$F(x) = F(0) + \frac{1}{2}f(x) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(y)\,dy$$

Same thing to solve for G...

$$2G'(x) = f'(x) + \frac{g(x)}{c}$$

$$G(x) = G(0) + \frac{1}{2}f(x) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^x g(y)\,dy$$

Solution is

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= F(0) + \frac{1}{2}f(x-ct) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy$$

$$+ G(0) + \frac{1}{2}f(x+ct) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^{x+ct} g(y)\,dy$$

Evaluate again at $t=0$

$$u(x,0) = F(0) + \frac{1}{2}f(x) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(y)\,dy$$

$$+ G(0) + \frac{1}{2}f(x) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^x g(y)\,dy = f(x)$$

Thus $F(0) + G(0) - f(0) + \cancel{f(x)} = \cancel{f(x)}$

$$F(0) + G(0) = f(0)$$

and

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= \cancel{F(0)} + \frac{1}{2}f(x-ct) - \cancel{\frac{1}{2}f(0)} - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy$$

$$+ \cancel{G(0)} + \frac{1}{2}f(x+ct) - \cancel{\frac{1}{2}f(0)} + \frac{1}{2c}\int_0^{x+ct} g(y)\,dy$$

Final solution

$$\boxed{u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy + \frac{1}{2c}\int_0^{x+ct} g(t)\,dy}$$

to

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0.}$$

$$\boxed{u(x,0) = f(x),} \quad -\infty < x < \infty$$

$$\boxed{\frac{\partial u}{\partial t}(x,0) = g(x),} \quad -\infty < x < \infty.$$